

DUALITY ON VALUE SEMIGROUPS

PHILIPP KORELL, LAURA TOZZO, AND MATHIAS SCHULZE

ABSTRACT. We establish a combinatorial counterpart of the Cohen–Macaulay duality on fractional ideals on curve singularities. To this end we consider the class of so-called good semigroup ideals. Under suitable algebraic conditions it contains all value semigroup ideals of fractional ideals. We give an intrinsic definition of canonical good semigroup ideals and deduce a duality on good semigroup ideals. Canonical fractional ideals are then characterized by having a canonical value semigroup ideal. We prove that the Cohen–Macaulay duality and our good semigroup duality are compatible under taking values.

CONTENTS

| | |
|--|----|
| 1. Introduction | 1 |
| 2. Preliminaries | 3 |
| 2.1. Regular and fractional ideals | 3 |
| 2.2. Valuation rings | 4 |
| 3. One-dimensional Cohen–Macaulay rings | 6 |
| 3.1. Integral closure and value semigroups | 6 |
| 3.2. Value semigroups and localization | 9 |
| 3.3. Value semigroups and completion | 10 |
| 4. Semigroups | 12 |
| 4.1. Good semigroups and their ideals | 12 |
| 4.2. Length and distance | 15 |
| 5. Duality | 17 |
| 5.1. Canonical ideals | 18 |
| 5.2. Duality on good semigroups | 19 |
| 5.3. Duality of fractional ideals | 24 |
| References | 26 |

1. INTRODUCTION

Value semigroups of curve singularities have been studied intensively for several decades. Lejeune-Jalabert [LJ73] and Zariski [Zar86] proved independently that, as an invariant of irreducible complex plane curve singularities, the value semigroup is equivalent to the Puiseux characteristics and hence to the topological type. Kunz [Kun70] showed that an analytically irreducible and residually rational local ring R is Gorenstein if and only if its (numerical) value semigroup Γ_R is *symmetric*. Jäger [Jäg77] used the symmetry condition

Date: September 2, 2016.

1991 *Mathematics Subject Classification.* 14H20 (Primary), 13C14, 20M12 (Secondary).

Key words and phrases. curve singularity, value semigroup, canonical module, duality.

The research leading to these results has received funding from the People Programme (Marie Curie Actions) of the European Union’s Seventh Framework Programme (FP7/2007-2013) under REA grant agreement n° PCIG12-GA-2012-334355.

to define a semigroup ideal K^0 such that (suitably normalized) canonical fractional ideals \mathcal{K} of R are characterized by having value semigroup ideal $\Gamma_{\mathcal{K}} = K^0$.

García [Gar82] was the first to describe a symmetry property of non-numerical semigroups; he showed that value semigroup of plane curves with two branches are symmetric. Delgado [Del87] then made the step to general algebroid curves proving an analogue of Kunz's result. Later Campillo, Delgado and Kiyek [CDK94] relaxed the hypotheses to include analytically reduced and residually rational local rings R with infinite residue field.

D'Anna [D'A97] extended Jäger's approach under the preceding hypotheses. He turned Delgado's symmetry definition into an explicit formula for a semigroup ideal K^0 (see Definition 5.2.1) such that any (suitably normalized) fractional ideal \mathcal{K} of R is canonical if and only if $\Gamma_{\mathcal{K}} = K^0$. In the process he studied axioms satisfied by value semigroup ideals which lead to the notion of a *good semigroup ideal* (see Definition 4.1.1).

Barucci, D'Anna and Fröberg [BDF00] studied some more special classes of rings like Gorenstein rings, Arf rings and rings of small multiplicity in relation with their value semigroups. Their setup includes the case of semilocal rings. Notably they found an example of a good semigroup which is not the value semigroup of any ring.

Recently Pol [Pol15a, Pol15b, Thm. 2.4] gave an explicit formula for the value semigroup ideal of the dual of a fractional ideal for Gorenstein algebroid curves.

In this paper, we extend and unify D'Anna's and Pol's results for a general class of rings R that we call *admissible* (see Definition 3.1.4). We give a simple definition of a *canonical semigroup ideal* K of a good semigroup (without any normalization) (see Definition 5.2.5). We show that it is equivalent to K inducing a duality $E \mapsto K - E$ on good semigroup ideals (see Theorem 5.2.7). This means that

$$K - (K - E) = E$$

for all good semigroup ideals. It turns out that our canonical semigroup ideals are exactly the translations of D'Anna's K^0 . In particular, D'Anna's characterization of canonical ideals in terms of their value semigroup ideals persists for admissible rings (see Corollary 5.3.6). We show that

$$\Gamma_{\mathcal{K}:\mathcal{E}} = \Gamma_{\mathcal{K}} - \Gamma_{\mathcal{E}}.$$

for any regular fractional ideal \mathcal{E} of R (see Theorem 5.3.5). This means that there is a commutative diagram

$$\begin{array}{ccc} \{\text{regular fractional ideals of } R\} & \xrightarrow{\mathcal{E} \mapsto \mathcal{K}:\mathcal{E}} & \{\text{regular fractional ideals of } R\} \\ \mathcal{E} \mapsto \Gamma_{\mathcal{E}} \downarrow & \circlearrowleft & \downarrow \mathcal{E} \mapsto \Gamma_{\mathcal{E}} \\ \{\text{good semigroup ideals of } \Gamma_R\} & \xrightarrow{E \mapsto \Gamma_{\mathcal{K}} - E} & \{\text{good semigroup ideals of } \Gamma_R\} \end{array}$$

relating the Cohen–Macaulay duality $\mathcal{E} \mapsto \mathcal{K} : \mathcal{E}$ on R to our good semigroup duality $E \mapsto K - E$ on Γ_R for $K = \Gamma_{\mathcal{K}}$.

An important tool to prove the commutativity of the above diagram is the *distance* $d(E \setminus F)$ between two good semigroup ideals $E \subset F$ (see Definition 4.2.3). It plays the role of the length $\ell(\mathcal{E}/\mathcal{F})$ of the quotient of two fractional ideals $\mathcal{E} \subset \mathcal{F}$ on the semigroup side. In fact, the two quantities agree in case $E = \Gamma_{\mathcal{E}}$ and $F = \Gamma_{\mathcal{F}}$ (see Proposition 4.2.8), that is,

$$\ell_R(\mathcal{F}/\mathcal{E}) = d(\Gamma_{\mathcal{F}} \setminus \Gamma_{\mathcal{E}}).$$

D'Anna [D'A97, Prop. 2.7] stated that $d(E \setminus F) = 0$ is equivalent to $E = F$, which implies $\mathcal{E} = \mathcal{F}$ in the preceding case. We give a proof of this crucial fact (see Proposition 4.2.6).

Before approaching these main results, we review the definition of value semigroups and their ideals and give a detailed account of their compatibility with localization and completion (see §3).

2. PRELIMINARIES

All rings under consideration will be commutative and unitary. We denote by $\text{Max}(R)$ the set of maximal ideals of a ring R .

The *total ring of fractions* Q_R of a ring R is the localization of R at the set R^{reg} of all regular elements of R . More generally, we set $S^{\text{reg}} := S \cap Q_R^{\text{reg}}$ for any subset $S \subset Q_R$. Note that $R^{\text{reg}} = R \cap Q_R^{\text{reg}}$.

We fix a ring Q with

$$(2.1) \quad Q^{\text{reg}} = Q^*$$

and abbreviate $\mathcal{F} : \mathcal{E} := \mathcal{F} :_Q \mathcal{E}$ for any subsets $\mathcal{E}, \mathcal{F} \subset Q$.

2.1. Regular and fractional ideals. Let R be a ring with $Q_R = Q$. Regular fractional ideals will play a central role in our considerations.

Definition 2.1.1.

- (a) An R -submodule \mathcal{E} of Q is called *regular* if $\mathcal{E}^{\text{reg}} \neq \emptyset$ or, equivalently, $Q\mathcal{E} = Q$.
- (b) An R -submodule $\mathcal{E} \subset Q$ such that $r\mathcal{E} \subset R$ for some $r \in R^{\text{reg}}$ is called a *fractional ideal* (of R). If R is Noetherian, this is equivalent to \mathcal{E} being a finitely generated R -submodule of Q .
- (c) If every regular ideal, or equivalently regular fractional ideal, I of R is generated by I^{reg} , then R is called a *Marot ring*.
- (d) The *conductor* of a fractional ideal \mathcal{E} of R is $\mathcal{C}_{\mathcal{E}} = \mathcal{E} : \overline{R}$, where \overline{R} denotes the integral closure of R in Q .

The set \mathfrak{R}_R of regular fractional ideals of R is a (commutative) monoid under product of ideals and closed under ideal quotient. We call an R -submodule \mathcal{E} of Q *invertible* if $\mathcal{E}\mathcal{F} = R$ for some R -submodule \mathcal{F} of Q which then is uniquely determined as $\mathcal{F} = \mathcal{E}^{-1} = R : \mathcal{E}$. Every invertible R -submodule of Q is regular and finitely generated (see [KV04, Ch. II, Rem. 2.1.(3) and Prop. 2.2.(1),(2)]). In particular, the (abelian) group \mathfrak{R}_R^* of all invertible R -submodule of Q is a submonoid of \mathfrak{R}_R . In case R is (quasi)semilocal, all elements of \mathfrak{R}_R^* are principal fractional ideals (see [KV04, Ch. II, Prop. 2.2.(3)]).

Remark 2.1.2. Let $x \in Q^{\text{reg}}$.

- (a) For any $\mathcal{E} \in \mathfrak{R}_R$, $x\mathcal{E} \in \mathfrak{R}_R$.
- (b) For any two $\mathcal{E}, \mathcal{F} \in \mathfrak{R}_R$, $(x\mathcal{E}) : \mathcal{F} = x(\mathcal{E} : \mathcal{F})$ and $\mathcal{E} : (x\mathcal{F}) = x^{-1}(\mathcal{E} : \mathcal{F})$.
- (c) For any two inclusions $\mathcal{E} \subset \mathcal{E}'$ and $\mathcal{F} \subset \mathcal{F}'$ of regular fractional ideals of R , $\mathcal{E} : \mathcal{F}' \subset \mathcal{E} : \mathcal{F} \subset \mathcal{E}' : \mathcal{F}$.

For $\mathcal{E}, \mathcal{F} \in \mathfrak{R}_R$ there is a natural isomorphism

$$\mathcal{F} : \mathcal{E} \rightarrow \text{Hom}_R(\mathcal{E}, \mathcal{F})$$

of R -modules (see [HK71, Lem. 2.1]) compatible with multiplication in Q and composition of homomorphisms. The composed isomorphism

$$\mathcal{F} : (\mathcal{F} : \mathcal{E}) \rightarrow \text{Hom}_R(\mathcal{F} : \mathcal{E}, \mathcal{F}) \rightarrow \text{Hom}_R(\text{Hom}_R(\mathcal{E}, \mathcal{F}), \mathcal{F})$$

fits into a commutative diagram of natural maps

$$(2.2) \quad \begin{array}{ccc} & & \mathcal{F} : (\mathcal{F} : \mathcal{E}) \\ & \nearrow & \downarrow \\ \mathcal{E} & & \text{Hom}_R(\text{Hom}_R(\mathcal{E}, \mathcal{F}), \mathcal{F}). \\ & \searrow & \end{array}$$

Given another ring S with $Q_S = Q$, a ring homomorphism $R \rightarrow S$ and fractional ideals \mathcal{E} of R and \mathcal{F} of S respectively, we have

$$(2.3) \quad \mathcal{E} : \mathcal{F} = (\mathcal{E} : S) : \mathcal{F}.$$

Lemma 2.1.3. *Let $R = (R, \mathfrak{m})$ be a local ring with \mathfrak{m} -adic completion \widehat{R} , and let \mathcal{E} and \mathcal{F} be R -submodules of Q . Then*

- (a) $R \rightarrow \widehat{R}$ is faithfully flat,
- (b) $\widehat{\mathcal{E}} = \widehat{R} \otimes_R \mathcal{E} = \widehat{R}\mathcal{E}$,
- (c) $\widehat{\mathcal{E}} \cap Q = \mathcal{E}$ and
- (d) $\widehat{\mathcal{E} \cap \mathcal{F}} = \widehat{\mathcal{E}} \cap \widehat{\mathcal{F}}$.

Proof. See [Mat89, Thm. 8.14] and [Bou61, Ch. 1, §3, Prop. 10]. \square

Lemma 2.1.4. *Let $R = (R, \mathfrak{m})$ be a one-dimensional local Cohen–Macaulay ring with \mathfrak{m} -adic completion \widehat{R} .*

- (a) $Q_{\widehat{R}} = \widehat{Q}$.
- (b) *Let $R \subset S \subset Q$ be an R -finite extension ring. Then there is a group isomorphism*

$$\mathfrak{R}_S \rightarrow \mathfrak{R}_{\widehat{S}}, \quad \mathcal{E} \mapsto \widehat{\mathcal{E}}, \quad \mathcal{F} \cap Q \mapsto \mathcal{F}.$$

Proof.

- (a) See [KV04, Ch. II, (2.4)].

(b) Since S is R -finite, also \widehat{S} is \widehat{R} -finite, and hence $\mathfrak{R}_S \subset \mathfrak{R}_R$ and $\mathfrak{R}_{\widehat{S}} \subset \mathfrak{R}_{\widehat{R}}$. For $S = R$ the isomorphism in question was established in [HK71, Lem. 2.11]. It remains to show that it induces the desired isomorphism. For any $\mathcal{E} \in \mathfrak{R}_S$, $\widehat{\mathcal{E}} = \mathcal{E}\widehat{R} = \mathcal{E}\widehat{R}S = \mathcal{E}\widehat{S}$ is an \widehat{S} -module, and hence $\widehat{\mathcal{E}} \in \mathfrak{R}_{\widehat{S}}$. Moreover, with $\mathcal{F} \in \mathfrak{R}_{\widehat{S}}$ also $\mathcal{F} \cap Q$ is an S -module, and therefore $\mathcal{F} \cap Q \in \mathfrak{R}_S$. \square

Lemma 2.1.5. *Let $R = (R, \mathfrak{m})$ be a local ring with $|R/\mathfrak{m}| \geq |\text{Max}(S)|$, where $R \subset S \subset Q$ is a semilocal extension ring, and let $\mathcal{E} \in \mathfrak{R}_R$ such that $\mathcal{E}S$ is principal. Then $\mathcal{E}S = xS$ for some $x \in \mathcal{E}^{\text{reg}}$. In particular, $R \subset y\mathcal{E} \subset S$ for $y = x^{-1} \in Q^{\text{reg}}$.*

Proof. See [Jäg77, Hilfssatz 2]. \square

2.2. Valuation rings. To deal with rings with zero-divisors, we need a general notion of valuation (ring), sometimes called a *Manis* or *pseudo-valuation (ring)* (see [KV04, Mat73, CDK94]). In case of one-dimensional Cohen–Macaulay rings only discrete valuation rings arise (see § 3 below).

Let Q be a ring satisfying (2.1) with a large Jacobson radical, that is, every prime ideal of Q containing the Jacobson radical of Q is a maximal ideal (see [KV04, Ch. I, Prop. 1.9] for equivalent characterizations). For example, any (quasi)semilocal ring has a large Jacobson radical. Under this assumption Q as well as every subring $R \subset Q$ with $Q_R = Q$ is a Marot ring (see [KV04, Ch. I, Prop. 1.12]).

Definition 2.2.1. A *valuation ring* of Q is a subring $V \subsetneq Q$ such that the set $Q \setminus V$ is multiplicatively closed. For any ring $R \subset V$ satisfying $Q_R = Q$, we call V a *valuation ring over R* . If $R \subset Q$ is a subring with $Q_R = Q$, we denote by \mathfrak{V}_R the set of all valuation rings of Q over R .

Remark 2.2.2. Let V be a valuation ring of Q .

- (a) V has a unique regular maximal ideal \mathfrak{m}_V . In particular, $V^{\text{reg}} \setminus V^* \subset \mathfrak{m}_V$ (see [KV04, Ch. I, Thm. 2.2]).
- (b) If $A \subset Q$ is a subring strictly containing V , then $A = Q$ (see [KV04, Ch. I, Prop. 2.15.(d)]).
- (c) Each $\mathcal{E} \in \mathfrak{R}_V^*$ is principal (see [KV04, Ch. II, Prop. 2.2.(2) and Ch. I, Prop. 2.4.(2)]).

For a valuation ring V of Q , the group \mathfrak{R}_V^* is totally ordered by reverse inclusion (see [KV04, Ch. I, Thm. 2.2]). The *infinite prime ideal* of V

$$I_V := V : Q = \bigcap_{\mathcal{E} \in \mathfrak{R}_V^*} \mathcal{E} \in \text{Spec}(V) \cap \text{Spec}(Q)$$

is the intersection of all regular (principal) fractional ideals of V (see [KV04, Ch. I, Prop. 2.4.3.a]). We include \mathfrak{R}_V^* into the totally ordered monoid $\mathfrak{R}_{V,\infty}^* = \mathfrak{R}_V^* \cup \{I_V\}$. For $\mathcal{E}, \mathcal{F} \in \mathfrak{R}_{V,\infty}^*$ we have $\mathcal{E}\mathcal{F} = I_V$ if $\{\mathcal{E}, \mathcal{F}\} \not\subset \mathfrak{R}_V^*$, and $\mathcal{E} < I_V$ for all $\mathcal{E} \in \mathfrak{R}_V^*$.

For $x \in Q$, we denote by $\mu_V(x)$ the intersection of all regular V -submodules of Q containing x . If $x \in Q \setminus I_V$, then $\mu_V(x) \in \mathfrak{R}_V^*$ (see [KV04, Ch. I, Prop. 2.4.3.b]). In particular, $\mu_V(x) = xV$ if $x \in Q^{\text{reg}}$, and $\mu_V(x) = I_V$ if and only if $x \in I_V$. This yields a map

$$\mu_V : Q \rightarrow \mathfrak{R}_{V,\infty}^*$$

satisfying (see [KV04, Ch. I, Prop. 2.13])

- (V1) $\mu_V(xy) = \mu_V(x)\mu_V(y)$ and
- (V2) $\mu_V(x + y) \geq \min\{\mu_V(x), \mu_V(y)\}$

for any $x, y \in Q$. Note that equality holds in (V2) if $\mu_V(x) \neq \mu_V(y)$. We can write

$$V = \{x \in Q \mid \mu_V(x) \geq V\}$$

with regular maximal ideal (see Remark 2.2.2.(a))

$$\mathfrak{m}_V = \{x \in Q \mid \mu_V(x) > V\}.$$

If the regular maximal ideal \mathfrak{m}_V of a valuation ring V is finitely generated, then by [KV04, Ch. I, Prop. 2.4.(2)]

$$\mathfrak{m}_V = \min\{\mathcal{E} \in \mathfrak{R}_V^* \mid V < \mathcal{E}\} \in \mathfrak{R}_V^*.$$

Definition 2.2.3. A valuation ring V of Q is called a *discrete valuation ring* if its regular maximal ideal \mathfrak{m}_V is finitely generated (see [KV04, Ch. I, Prop. 2.15] for equivalent conditions).

Let V be a discrete valuation ring of Q . Then there is a unique order preserving group isomorphism

$$(2.4) \quad \phi_V : \mathfrak{R}_V^* \xrightarrow{\cong} \mathbb{Z}, \quad \mathcal{E} \mapsto \phi(\mathcal{E}) = \max\{k \mid \mathfrak{m}_V^k \leq \mathcal{E}\}, \quad \mathfrak{m}_V^k \leftarrow k.$$

In fact, for $\mathcal{E} \in \mathfrak{R}_V^*$ and k maximal with $\mathfrak{m}_V^k \leq \mathcal{E}$, we have $V = \mathfrak{m}_V^k : \mathfrak{m}_V^k \leq \mathcal{E} : \mathfrak{m}_V^k < \mathfrak{m}_V$, and hence $\mathcal{E} = \mathfrak{m}_V^k$. Embedding \mathbb{Z} into the totally ordered monoid $\mathbb{Z}_\infty := \mathbb{Z} \cup \{\infty\}$ and

extending ϕ_V by setting $\phi_V(I_V) := \infty$ yields a commutative diagram

$$(2.5) \quad \begin{array}{ccc} Q_R & & \\ \mu_V \downarrow & \searrow \nu_V & \\ \mathfrak{R}_{V,\infty}^* & \xrightarrow[\phi_V]{\cong} & \mathbb{Z}_\infty. \end{array}$$

Definition 2.2.4. A *discrete valuation* of Q is a map $\nu: Q \rightarrow \mathbb{Z}_\infty$ satisfying conditions analogous to (V1) (considered additively in \mathbb{Z}_∞) and (V2). We refer to $\nu(x) \in \mathbb{Z}_\infty$ as the *value* of $x \in Q$ with respect to ν . The subring $V_\nu = \{x \in Q \mid \nu(x) \geq 0\}$ of Q is called the *valuation ring* of ν .

The above considerations show that $V \mapsto \nu_V$ and $\nu \mapsto V_\nu$ define a one-to-one correspondence between discrete valuation rings and discrete valuations of Q .

3. ONE-DIMENSIONAL COHEN–MACAULAY RINGS

3.1. Integral closure and value semigroups. If R is local and integrally closed, then it is a discrete valuation ring (see [KV04, Ch. II, Prop. 2.5]). In general, the totality \mathfrak{V}_R of valuation rings over R is described in the following theorem. This provides the foundation for defining and studying value semigroup ideals.

Theorem 3.1.1. *Let R be a one-dimensional semilocal Cohen–Macaulay ring with total ring of fractions $Q = Q_R$.*

- (a) *The set \mathfrak{V}_R is finite and non-empty, and it contains discrete valuation rings only.*
- (b) *$\text{Max}(Q) = \{I_V \mid V \in \mathfrak{V}_R\}$, and for any $I \in \text{Max}(Q)$, there is a bijection*

$$\{V \in \mathfrak{V}_R \mid I_V = I\} \leftrightarrow \mathfrak{V}_{R/(I \cap R)}, \quad V \mapsto V/I,$$

where $Q_{R/(I \cap R)} = Q/I$.

- (c) *The integral closure of R in Q can be written as $\overline{R} = \bigcap \mathfrak{V}_R$, its set of regular prime ideals agrees with $\text{Max}(\overline{R})$, and any regular ideal of \overline{R} is principal.*
- (d) *There is a bijection*

$$\text{Max}(\overline{R}) \leftrightarrow \mathfrak{V}_R, \quad \mathfrak{n} \mapsto ((\overline{R} \setminus \mathfrak{n})^{\text{reg}})^{-1} \overline{R}, \quad \mathfrak{n}_V := \mathfrak{m}_V \cap \overline{R} \leftrightarrow V.$$

In particular, $\overline{R}/\mathfrak{n}_V = V/\mathfrak{m}_V$.

Proof. See [KV04, Ch. II, Thm. 2.11]. □

By Theorem 3.1.1.(c) we have $\mathfrak{R}_{\overline{R}} = \mathfrak{R}_{\overline{R}}^*$, and there is a group isomorphism

$$\psi: \mathfrak{R}_{\overline{R}} \rightarrow \prod_{V \in \mathfrak{V}_R} \mathfrak{R}_V^*, \quad \mathcal{E} \mapsto (\mathcal{E}V)_{V \in \mathfrak{V}_R}, \quad \bigcap_{V \in \mathfrak{V}_R} \mathcal{E}V \hookleftarrow (\mathcal{E}V)_{V \in \mathfrak{V}_R}.$$

In fact, writing $\mathcal{E} = t\overline{R}$ for some $t \in Q^{\text{reg}}$,

$$\bigcap_{V \in \mathfrak{V}_R} \mathcal{E}V = \bigcap_{V \in \mathfrak{V}_R} tV = t \bigcap_{V \in \mathfrak{V}_R} V = t\overline{R} = \mathcal{E}$$

by Theorem 3.1.1.(c) and ψ is injective. Diagram (2.5) taken component-wise gives rise to a commutative diagram

$$(3.1) \quad \begin{array}{ccccc} & & Q^{\text{reg}} & & \\ & \swarrow & \downarrow \mu & \searrow \nu & \\ \mathfrak{R}_{\overline{R}} & \xrightarrow[\psi]{\cong} & \prod_{V \in \mathfrak{V}_R} \mathfrak{R}_V^* & \xrightarrow[\phi]{\cong} & \mathbb{Z}^{\mathfrak{V}_R}. \end{array}$$

Then surjectivity of μ , and hence of ψ , follows from Theorem 3.1.1.(d) and the Chinese Remainder Theorem (see the approximation theorem for discrete valuations in [KV04, Ch. I, Thm. 2.20.(3)]). The isomorphisms ψ and ϕ preserve the partial orders on $\mathfrak{R}_{\overline{R}}$ and $\prod_{V \in \mathfrak{V}_R} \mathfrak{R}_V^*$ by reverse inclusion and the natural partial order on $\mathbb{Z}^{\mathfrak{V}_R}$.

Definition 3.1.2. Let R be a one-dimensional semilocal Cohen–Macaulay ring, and let \mathfrak{V}_R be the set of (discrete) valuation rings of Q_R over R (see Theorem 3.1.1.(a)) with corresponding valuations

$$\nu = (\nu_V)_{V \in \mathfrak{V}_R} : Q_R \rightarrow \mathbb{Z}_{\infty}^{\mathfrak{V}_R}.$$

To each fractional ideal \mathcal{E} of R we associate its *value semigroup ideal*

$$\Gamma_{\mathcal{E}} := \nu(\mathcal{E}^{\text{reg}}) \subset \mathbb{Z}^{\mathfrak{V}_R}.$$

If $\mathcal{E} = R$, then the monoid Γ_R is called the *value semigroup* of R .

The following result was stated without proof in [BDF00, §2].

Proposition 3.1.3. *Let R be a one-dimensional semilocal Cohen–Macaulay ring with value semigroup Γ_R . Then the following are equivalent:*

- (i) *The ring R is local.*
- (ii) *The only element of Γ_R with a zero component in $\mathbb{Z}^{\mathfrak{V}_R}$ is 0.*

Proof.

(i) \Rightarrow (ii) See [Del88, (1.1.1)].

(ii) \Rightarrow (i) Suppose that 0 is the only element of Γ_R with a zero component in $\mathbb{Z}^{\mathfrak{V}_R}$, and set $\mathfrak{m} := \{x \in R \mid \nu(x) > 0\}$. Since $R^* = \overline{R}^* \cap R = \{z \in R \mid \nu(z) = 0\}$ (see Theorem 3.1.1.(b) and (c)), any proper ideal of R is contained in \mathfrak{m} . Moreover, \mathfrak{m} is obviously closed under multiplication by elements of R . We show that $\nu(x)$ has no zero component for all $x \in \mathfrak{m}$. This implies that \mathfrak{m} is also closed under addition, and hence an ideal.

For this, assume that there is $x \in \mathfrak{m}$ such that $\nu_{V_1}(x) = 0$ for some $V_1 \in \mathfrak{V}_R$. Then $x \in R \setminus R^{\text{reg}} \subset \bigcup_{V \in \mathfrak{V}_R} I_V$ by hypothesis on Γ_R and Theorem 3.1.1.(b). Thus, there is $V_1 \neq V_2 \in \mathfrak{V}_R$ such that $x \in I_{V_2}$.

Choose $y \in R^{\text{reg}} \setminus R^*$. In particular, this means that $\nu(y) \neq 0$. After replacing y by a suitable power, we may assume that $\nu_V(x) \neq \nu_V(y)$ for all $V \in \mathfrak{V}_R$. Then $\nu(x + y) = \min\{\nu(x), \nu(y)\} \in \mathbb{Z}^{\mathfrak{V}_R}$ by (V2) and since $\nu(y) \in \nu(Q^{\text{reg}}) = \mathbb{Z}^{\mathfrak{V}_R}$. Thus, $x + y \in R^{\text{reg}}$ again since $R \setminus R^{\text{reg}} \subset \bigcup_{V \in \mathfrak{V}_R} I_V$, and hence $\nu(x + y) \in \Gamma_R$.

Therefore, by assumption on Γ_R , $\nu_{V_1}(x + y) = \nu_{V_1}(x) = 0$ yields $\nu(x + y) = 0$, and thus $\nu_{V_2}(y) = \nu_{V_2}(x + y) = 0$ implies $\nu(y) = 0$, contradicting the choice of y . \square

In the following we will show that, under suitable hypotheses, semigroups $E = \Gamma_{\mathcal{E}}$ of fractional ideals \mathcal{E} of R have certain properties used to define the notion of a good semigroup in §4.

Definition 3.1.4. Let R be a one-dimensional semilocal Cohen–Macaulay ring.

- (a) Then R is called *residually rational* if $R/\mathfrak{m} = \overline{R}/\mathfrak{n}$ for any $\mathfrak{m} \in \text{Max}(R)$ and $\mathfrak{n} \in \text{Max}(\overline{R})$ with $\mathfrak{n} \cap R = \mathfrak{m}$. Equivalently, $R/\mathfrak{m} = V/\mathfrak{m}_V$ for any $\mathfrak{m} \in \text{Max}(R)$ and $V \in \mathfrak{V}_R$ with $\mathfrak{m}_V \cap R = \mathfrak{m}$ (see Theorem 3.1.1.(d)).
- (b) We say that R has *large residue fields* if $|R/\mathfrak{m}| \geq |\mathfrak{V}_{R\mathfrak{m}}|$ for all $\mathfrak{m} \in \text{Max}(R)$.
- (c) We call R *admissible* if it is analytically reduced and residually rational with large residue fields.

Definition 3.1.5. Let \overline{S} be a partially ordered monoid, isomorphic to \mathbb{N}^I with its natural partial order, where I is a finite set. We consider the following properties of a subset E of the group of differences $D_{\overline{S}} \cong \mathbb{Z}^I$ of \overline{S} (see [Del88, §1] and [D'A97, §2]).

- (E0) There exists an $\alpha \in D_{\overline{S}}$ such that $\alpha + \overline{S} \subset E$.
- (E1) If $\alpha, \beta \in E$, then $\min\{\alpha, \beta\} := (\min\{\alpha_i, \beta_i\})_{i \in I} \in E$.
- (E2) For any $\alpha, \beta \in E$ and $j \in I$ with $\alpha_j = \beta_j$ there exists an $\epsilon \in E$ such that $\epsilon_j > \alpha_j = \beta_j$ and $\epsilon_i \geq \min\{\alpha_i, \beta_i\}$ for all $i \in I \setminus \{j\}$ with equality if $\alpha_i \neq \beta_i$.

Lemma 3.1.6. Any group automorphism φ of \mathbb{Z}^s preserving the partial order is defined by a permutation of the standard basis.

Proof. Let φ be an automorphism of \mathbb{Z}^s preserving the partial order. Then $(\varphi(\mathbf{e}_i))_{i \in \{1, \dots, s\}}$ is a basis of \mathbb{Z}^s , and hence $0 < \mathbf{e}_j = \sum_i \lambda_i \varphi(\mathbf{e}_i) = \varphi(\sum_i \lambda_i \mathbf{e}_i)$ for some $\lambda_i \in \mathbb{Z}$. Since φ is order preserving, this implies $\lambda_i \in \mathbb{N}$ for all i . It follows that $\mathbf{e}_j = \varphi(\mathbf{e}_i)$ for some i . \square

Lemma 3.1.7. Let R be a one-dimensional semilocal Cohen–Macaulay ring. If R is analytically reduced, then $\mathcal{C}_{\mathcal{E}} \in \mathfrak{R}_R \cap \mathfrak{R}_{\overline{R}}$ for any $\mathcal{E} \in \mathfrak{R}_R$. In particular, $\mathcal{C}_{\mathcal{E}} = x\overline{R}$ for some $x \in \mathcal{C}_{\mathcal{E}}^{\text{reg}}$.

Proof. The ring R is analytically reduced if and only if its normalization \overline{R} is a finite R -module (see [KV04, Ch. II, Thm. 3.22]). This implies $\overline{R} \in \mathfrak{R}_R$, and therefore $\mathcal{C}_{\mathcal{E}} = \mathcal{E} : \overline{R}$ is a regular fractional ideal of R (see §2.1). Since $\mathcal{C}_{\mathcal{E}} \in \mathfrak{R}_{\overline{R}}$, Theorem 3.1.1.(c) yields $\mathcal{C}_{\mathcal{E}} = x\overline{R}$ for some $x \in \mathcal{C}_{\mathcal{E}}^{\text{reg}}$. \square

In the following, we collect results from [D'A97] and provide a detailed proof.

Proposition 3.1.8. Let R be a one-dimensional semilocal Cohen–Macaulay ring with value semigroup $S := \Gamma_R$, and let $E := \Gamma_{\mathcal{E}}$ for some $\mathcal{E} \in \mathfrak{R}_R$.

- (a) We have $E + S \subset E$.
- (b) If R is analytically reduced, then E satisfies (E0).
- (c) If R is local with large residue field, then E satisfies (E1).
- (d) If R is local and residually rational, then E satisfies (E2).

Proof. (a) This follows from ν in Diagram (3.1) being a group homomorphism.

(b) By Lemma 3.1.7 there is $x \in \mathcal{C}_{\mathcal{E}}^{\text{reg}}$ such that

$$\nu(x) + \mathbb{N}^{\mathfrak{V}_R} = \nu(x\overline{R}^{\text{reg}}) = \nu(\mathcal{C}_{\mathcal{E}}^{\text{reg}}) \subset \nu(\mathcal{E}^{\text{reg}}) = E.$$

(c) Let $x, y \in \mathcal{E}^{\text{reg}}$ with $\nu(x) = \alpha$ and $\nu(y) = \beta$. By Lemma 2.1.5, we may assume that $\langle x, y \rangle_{\overline{R}} = z\overline{R}$ for some $z \in \langle x, y \rangle_R^{\text{reg}} \subset \mathcal{E}^{\text{reg}}$. Then $\nu(z) \geq \min\{\nu(x), \nu(y)\} \geq \nu(z)$ by Definition 2.2.4, and hence $\min\{\nu(x), \nu(y)\} = \nu(z) \in E$.

(d) Denote by \mathfrak{m} be the maximal ideal of R . Let $\alpha, \beta \in E$ and $W \in \mathfrak{V}_R$ such that $\alpha_W = \beta_W$. Pick $x, y \in \mathcal{E}^{\text{reg}}$ such that $\nu(x) = \alpha$ and $\nu(y) = \beta$. Then $\nu_W(x/y) = \alpha_W - \beta_W = 0$, and hence $x/y \in (W/\mathfrak{m}_W)^*$. By Theorem 3.1.1.(d) and hypothesis, $(R/\mathfrak{m})^* = (V/\mathfrak{m}_V)^*$ for all $V \in \mathfrak{V}_R$. It follows that there exists $u \in R \setminus \mathfrak{m}$ such that $x/y = \overline{u} \in R/\mathfrak{m}$ and, in particular, $\nu(u) = 0$. Then $uy - x \in \mathcal{E}$ with $\nu_W(uy - x) = \nu_W(u - x/y) + \nu_W(y) > \nu_W(y) = \beta_W$ and $\nu_V(uy - x) \geq \min\{\alpha_V, \beta_V\}$ for all $V \in \mathfrak{V}_R \setminus \{W\}$ with equality if $\alpha_V \neq \beta_V$. This

remains true after replacing u by any element $u' \in u + \mathfrak{m}$. It is left to show that, for some u , $\nu_V(u - x/y) < \infty$ for all $V \in \mathfrak{V}_R$ with $\alpha_V = \beta_V$. Since R is Cohen–Macaulay, there is an $m \in \mathfrak{m}^{\text{reg}} \subset \mathfrak{m}_W^{\text{reg}}$, and hence $(\infty, \dots, \infty) > \nu(m^k) \geq k \cdot (1, \dots, 1)$. Then any $u' = u + m^k$ with $k \geq \max\{\nu_V(u - x/y) \neq \infty \mid V \in \mathfrak{V}_R \text{ with } \alpha_V = \beta_V\}$ satisfies the requirement. \square

3.2. Value semigroups and localization. Let R be a reduced semilocal ring. Since R is reduced, we have $Q_R = \prod_{\mathfrak{p} \in \text{Min}(R)} Q_{R/\mathfrak{p}}$. In particular, $Q_{R_{\mathfrak{m}}} = \prod_{\mathfrak{m} \supset \mathfrak{p} \in \text{Min}(R)} Q_{R_{\mathfrak{m}}/\mathfrak{p}R_{\mathfrak{m}}}$ for any $\mathfrak{m} \in \text{Max}(R)$. If R is a domain and $\mathfrak{p} \in \text{Spec}(R)$, then $R_{\mathfrak{p}} \subset Q_R$, and hence $Q_{R_{\mathfrak{p}}} = Q_R$. It follows that $Q_{R_{\mathfrak{m}}} = \prod_{\mathfrak{m} \supset \mathfrak{p} \in \text{Min}(R)} Q_{R/\mathfrak{p}}$, and thus $Q_{R_{\mathfrak{m}}} = (Q_R)_{\mathfrak{m}}$.

Lemma 3.2.1. *Let R be a reduced one-dimensional semilocal Cohen–Macaulay ring.*

- (a) *For any $V \in \mathfrak{V}_R$ we have $I_V = \prod_{\mathfrak{q}_V \neq \mathfrak{p} \in \text{Min}(R)} Q_{R/\mathfrak{p}} \times 0$, where $\mathfrak{q}_V := I_V \cap R \in \text{Min}(R)$.*
- (b) *For any $\mathfrak{q} \in \text{Min}(R)$ there is a bijection*

$$\{V \in \mathfrak{V}_R \mid \mathfrak{q}_V = \mathfrak{q}\} \rightarrow \mathfrak{V}_{R/\mathfrak{q}}, \quad V \mapsto V/I_V, \quad \prod_{\mathfrak{q} \neq \mathfrak{p} \in \text{Min}(R)} Q_{R/\mathfrak{p}} \times \overline{V} \leftarrow \overline{V}.$$

- (c) *If R is a domain, $\mathfrak{m} \in \text{Spec}(R)$ and $V \in \mathfrak{V}_R$, then $\mathfrak{m} \in \text{Max}(R)$ with $V \in \mathfrak{V}_{R_{\mathfrak{m}}}$ if and only if $\mathfrak{m} = \mathfrak{m}_V \cap R$.*
- (d) *For any $V \in \mathfrak{V}_R$ we have $\mathfrak{q}_V \subset \mathfrak{m}_V \cap R \in \text{Max}(R)$.*

Proof. (a) This follows since $I_V \in \text{Max}(Q)$ due to Theorem 3.1.1.(b).

(b) This follows from the bijection in Theorem 3.1.1.(b) (see [KV04, Ch. II, 2.12]).

(c) First note that $\mathfrak{m}_V \cap R \in \text{Max}(R)$. Otherwise $\mathfrak{m}_V \cap R = 0$ since R is a one-dimensional domain. This implies $R^{\text{reg}} \subset V^*$, and hence $Q_R \subset V$ in contradiction to $V \in \mathfrak{V}_R$. Now $R \setminus \mathfrak{m} \subset V \setminus \mathfrak{m}_V$ implies $R_{\mathfrak{m}} \subset V \subset Q_R = Q_{R_{\mathfrak{m}}}$, and therefore $V \in \mathfrak{V}_{R_{\mathfrak{m}}}$. Conversely, let $V \in \mathfrak{V}_{R_{\mathfrak{m}}}$, and assume that $\mathfrak{m}_V \cap R \neq \mathfrak{m} \in \text{Max}(R)$. Then $(\mathfrak{m}_V \cap R) \setminus \mathfrak{m} \subset (R_{\mathfrak{m}})^* \subset V^*$ which is a contradiction.

(d) By (a) we have $\mathfrak{q}_V \subset \mathfrak{m}_V \cap R$. Since $\mathfrak{m}_V = \prod_{\mathfrak{q}_V \neq \mathfrak{p} \in \text{Min}(R)} Q_{R/\mathfrak{p}} \times \mathfrak{m}_{\overline{V}}$ by (b), $(\mathfrak{m}_V \cap R)/\mathfrak{q}_V = \mathfrak{m}_{\overline{V}} \cap (R/\mathfrak{q}_V)$. Then (c) yields the claim. \square

Let R be a reduced one-dimensional semilocal Cohen–Macaulay ring. For any $\mathfrak{m} \in \text{Max}(R)$, Lemma 3.2.1.(b) gives rise to a natural bijection

$$\chi_{\mathfrak{m}}: \{V \in \mathfrak{V}_R \mid \mathfrak{m}_V \cap R = \mathfrak{m}\} \rightarrow \mathfrak{V}_{R_{\mathfrak{m}}}, \quad V \mapsto V_{\mathfrak{m}}, \quad \prod_{\mathfrak{m} \not\supset \mathfrak{p} \in \text{Min}(R)} Q_{R/\mathfrak{p}} \times V' \leftarrow V'$$

since $\{\overline{V} \in \mathfrak{V}_{R/\mathfrak{q}_V} \mid \mathfrak{m}_{\overline{V}} \cap R/\mathfrak{q}_V = \mathfrak{m}(R/\mathfrak{q}_V)\} = \mathfrak{V}_{(R/\mathfrak{q}_V)_{\mathfrak{m}(R/\mathfrak{q}_V)}}$ by Lemma 3.2.1.(c).

The sets $\{V \in \mathfrak{V}_R \mid \mathfrak{m}_V \cap R = \mathfrak{m}\}$, $\mathfrak{m} \in \text{Max}(R)$, are clearly pairwise disjoint. By Lemma 3.2.1.(b) and (c) also the sets $\mathfrak{V}_{R_{\mathfrak{m}}}$, $\mathfrak{m} \in \text{Max}(R)$, are pairwise disjoint. Hence, we obtain a bijection

$$\rho: \mathfrak{V}_R \rightarrow \bigcup_{\mathfrak{m} \in \text{Max}(R)} \mathfrak{V}_{R_{\mathfrak{m}}}, \quad V \mapsto \chi_{\mathfrak{m}_V \cap R}(V) = V_{\mathfrak{m}_V \cap R}.$$

Using this, we define an order preserving group isomorphism

$$\xi: \prod_{V \in \mathfrak{V}_R} \mathfrak{R}_V^* \rightarrow \prod_{\mathfrak{m} \in \text{Max}(R)} \prod_{W \in \mathfrak{V}_{R_{\mathfrak{m}}}} \mathfrak{R}_W^*, \quad (\mathfrak{m}_V^{k_V})_{V \in \mathfrak{V}_R} \mapsto (\mathfrak{m}_{\rho(V)}^{k_V})_{V \in \mathfrak{V}_R}$$

Since for $V \in \mathfrak{V}_R$ we have $\mathfrak{m}_V = \prod_{\mathfrak{q}_V \neq \mathfrak{p} \in \text{Min}(R)} Q_{R/\mathfrak{p}} \times \mathfrak{m}_{\overline{V}}$, this yields an order preserving group isomorphism (see §3.1)

$$\vartheta: \mathfrak{R}_{\overline{R}} \rightarrow \prod_{\mathfrak{m} \in \text{Max}(R)} \mathfrak{R}_{\overline{R}_{\mathfrak{m}}}, \quad \mathcal{E} \mapsto \prod_{\mathfrak{m} \in \text{Max}(R)} \mathcal{E}_{\mathfrak{m}},$$

and we obtain a commutative diagram

$$\begin{array}{ccccccc} Q_R^{\text{reg}} & \xrightarrow{\quad} & \mathfrak{R}_{\overline{R}} & \xrightarrow{\cong} & \prod_{V \in \mathfrak{V}_R} \mathfrak{R}_V^* & \xrightarrow[\phi]{\cong} & \mathbb{Z}^{\mathfrak{V}_R} \\ \downarrow \iota & & \downarrow \vartheta \cong & & \downarrow \xi \cong & & \downarrow \sigma \cong \\ \prod_{\mathfrak{m} \in \text{Max}(R)} Q_{R_{\mathfrak{m}}}^{\text{reg}} & \xrightarrow{\quad} & \prod_{\mathfrak{m} \in \text{Max}(R)} \mathfrak{R}_{\overline{R}_{\mathfrak{m}}} & \xrightarrow{\cong} & \prod_{\mathfrak{m} \in \text{Max}(R)} \prod_{W \in \mathfrak{V}_{R_{\mathfrak{m}}}} \mathfrak{R}_W^* & \xrightarrow[(\phi_{\mathfrak{m}})_{\mathfrak{m}}]{\cong} & \prod_{\mathfrak{m} \in \text{Max}(R)} \mathbb{Z}^{\mathfrak{V}_{R_{\mathfrak{m}}}}. \end{array}$$

This implies $\nu(x) = (\nu_{\mathfrak{m}}(x))_{\mathfrak{m} \in \text{Max}(R)}$ for all $x \in Q_R^{\text{reg}}$.

Theorem 3.2.2. *Let R be a reduced one-dimensional semilocal Cohen–Macaulay ring, and let $\mathcal{E} \in \mathfrak{R}_R$. Then*

$$\Gamma_{\mathcal{E}} = \prod_{\mathfrak{m} \in \text{Max}(R)} \Gamma_{\mathcal{E}_{\mathfrak{m}}}.$$

Proof. Together with our considerations above the proof is in [BDF00, § 1.1]. \square

Corollary 3.2.3. *Let R be a reduced one-dimensional semilocal Cohen–Macaulay ring with large residue fields, and let $E = \Gamma_{\mathcal{E}}$ for some $\mathcal{E} \in \mathfrak{R}_R$.*

- (a) E satisfies (E1).
- (b) If R is residually rational, then E satisfies (E2).

Proof. Using Theorem 3.2.2, this follows from Proposition 3.1.8.(c) and (d). Note that to prove property (E2) for elements $\alpha, \beta \in \Gamma_{\mathcal{E}}$ which are different in all components in $\Gamma_{\mathcal{E}_{\mathfrak{m}}}$ for some $\mathfrak{m} \in \text{Max}(R)$ we need to apply (E1) in $\Gamma_{\mathcal{E}_{\mathfrak{m}}}$. \square

3.3. Value semigroups and completion. The compatibility of the semigroup ideals with completion is due to D’Anna (see [D’A97, §1]). We give a proof including the semilocal case. In the following, $\widehat{}$ stands for the completion at the Jacobson radical of R . Note that the Cohen–Macaulay property is invariant under completion (see [BH93, Cor. 2.1.8.(b)]).

Theorem 3.3.1. *Let R be a one-dimensional local Cohen–Macaulay ring with total ring of fractions $Q = Q_R$. Then there is a bijection*

$$\mathfrak{V}_R \rightarrow \mathfrak{V}_{\widehat{R}}, \quad V \mapsto \widehat{V}, \quad W \cap Q \leftarrow W.$$

Proof. See [KV04, Ch. II, Thm. 3.19.(2)]. \square

Corollary 3.3.2. *Let $R = (R, \mathfrak{m})$ be a local Cohen–Macaulay ring with \mathfrak{m} -adic completion \widehat{R} . Then $\widehat{\widehat{R}} = \overline{R}\widehat{R} = \widehat{\overline{R}}$.*

Proof. This follows from Lemma 2.1.3.(d) and Theorems 3.1.1.(c) and 3.3.1. \square

Let R be an analytically reduced one-dimensional local Cohen–Macaulay ring. Then \overline{R} is finite over R (see [KV04, Ch. II, Thm. 3.22]), and $\widehat{\widehat{R}}$ is finite over \widehat{R} . By Lemma 2.1.4.(b) there is an order preserving group isomorphism

$$\theta: \mathfrak{R}_{\overline{R}} \rightarrow \mathfrak{R}_{\widehat{\widehat{R}}}, \quad \mathcal{E} \mapsto \widehat{\mathcal{E}}, \quad \mathcal{F} \cap Q \leftarrow \mathcal{F}$$

Combining this with Diagram (3.1) for R and \widehat{R} , Theorem 3.3.1 and Corollary 3.3.2 then yield a commutative diagram

$$(3.2) \quad \begin{array}{ccccccc} Q_R^{\text{reg}} & & & & & & \\ \downarrow \iota & \searrow & & \xrightarrow{\nu} & & & \\ & \mathfrak{R}_{\overline{R}} & \xrightarrow[\psi]{\cong} & \prod_{V \in \mathfrak{V}_R} \mathfrak{R}_V^* & \xrightarrow[\phi]{\cong} & \mathbb{Z}^{\mathfrak{V}_R} & \\ & \downarrow \theta \cong & & \downarrow \eta \cong & & \downarrow \zeta \cong & \\ & \mathfrak{R}_{\widehat{R}} & \xrightarrow[\widehat{\psi}]{\cong} & \prod_{\widehat{V} \in \mathfrak{V}_{\widehat{R}}} \mathfrak{R}_{\widehat{V}}^* & \xrightarrow[\widehat{\phi}]{\cong} & \mathbb{Z}^{\mathfrak{V}_{\widehat{R}}} & \\ & \nearrow & & \nwarrow & & & \\ Q_{\widehat{R}}^{\text{reg}} & & & \xrightarrow{\widehat{\nu}} & & & \end{array}$$

where

$$\eta: \prod_{V \in \mathfrak{V}_R} \mathfrak{R}_V^* \rightarrow \prod_{\widehat{V} \in \mathfrak{V}_{\widehat{R}}} \mathfrak{R}_{\widehat{V}}^*, \quad (\mathcal{E}_V)_{V \in \mathfrak{V}_R} \mapsto (\widehat{\mathcal{E}}_V)_{V \in \mathfrak{V}_R}$$

is an isomorphism since ψ , θ and $\widehat{\psi}$ are isomorphisms.

Notation 3.3.3. For any R -submodule \mathcal{E} of Q_R , we define a decreasing filtration \mathcal{E}^\bullet on \mathcal{E} by

$$\mathcal{E}^\alpha := \{x \in \mathcal{E} \mid \nu(x) \geq \alpha\}$$

for any $\alpha \in \mathbb{Z}^{\mathfrak{V}_R}$. Note that with \mathcal{E} also \mathcal{E}^α is a (regular) fractional ideal.

Theorem 3.3.4. *Let R be an analytically reduced one-dimensional semilocal Cohen–Macaulay ring with large residue fields. Then*

$$\Gamma_{\mathcal{E}} = \Gamma_{\widehat{\mathcal{E}}}$$

for any $\mathcal{E} \in \mathfrak{R}_R$.

Proof. First suppose that R is local. By Diagram (3.2) and Lemma 2.1.3.(d), $\widehat{\mathcal{E}}^\alpha = (\widehat{\mathcal{E}})^\alpha$ for any $\alpha \in \mathbb{Z}^{\mathfrak{V}_R}$. Note that $\mathcal{E}^\alpha / \mathcal{E}^{\alpha + \mathbf{e}_V} \neq 0$ if and only if there is a $\beta \in \Gamma_{\mathcal{E}}$ with $\beta \geq \alpha$ and $\beta_V = \alpha_V$. By Proposition 3.1.8.(c), $\alpha \in \Gamma_{\mathcal{E}}$ is therefore equivalent to $\mathcal{E}^\alpha / \mathcal{E}^{\alpha + \mathbf{e}_V} \neq 0$ for all $V \in \mathfrak{V}_R$ (see [CDK94, Rem. 4.3]). This latter condition commutes with completion by Lemma 2.1.3.(a). In the general case we have

$$(3.3) \quad \widehat{R} = \prod_{\mathfrak{m} \in \text{Max}(R)} \widehat{R}_{\mathfrak{m}}$$

(see [Nag62, Thm. 17.7]), and the claim follows using Theorem 3.2.2. \square

Remark 3.3.5. Let R be an analytically reduced one-dimensional local Cohen–Macaulay ring. Then \widehat{R} is a reduced one-dimensional local Cohen–Macaulay ring.

By Theorem 3.3.1, Theorem 3.1.1.(d), Corollary 3.3.2, Equation (3.3), and since $\widehat{R}_{\mathfrak{m}}$ is a domain for any $\mathfrak{m} \in \text{Max}(\overline{R})$ (see [KV04, Ch. II, Prop. 2.5]), there is a sequence of bijections

$$\mathfrak{V}_R \leftrightarrow \mathfrak{V}_{\widehat{R}} \leftrightarrow \text{Max}(\overline{\widehat{R}}) \leftrightarrow \text{Max}(\widehat{\overline{R}}) \leftrightarrow \text{Min}(\widehat{\overline{R}}) \leftrightarrow \text{Min}(\overline{\widehat{R}}) \leftrightarrow \text{Min}(\widehat{R})$$

sending V to $\mathfrak{q}_{\widehat{V}}$. If in addition $R = \widehat{R}$, then $V/I_V = \overline{R/\mathfrak{q}_V}$ by Lemma 3.2.1.(b) and Theorem 3.1.1.(c). Moreover, $\nu_V = \nu_{R/\mathfrak{q}_V} \circ \pi_V$, where $\pi_V: Q \twoheadrightarrow Q_{R/\mathfrak{q}_V} = Q/I_V$ (see Lemma 3.2.1.(a)) since the natural map $\mathfrak{R}_V^* \rightarrow \mathfrak{R}_{V/I_V}^*$ is a group isomorphism. Therefore, the product

$$(\nu_{R/\mathfrak{q}_V})_{V \in \mathfrak{V}_R}: Q = \prod_{V \in \mathfrak{V}_R} Q_{R/\mathfrak{q}_V} \rightarrow \mathbb{Z}^{\mathfrak{V}_R}$$

yields the same semigroup as in Definition 3.1.2. This approach is often used in the literature (see [KW84, Del87, Del88, D'A97]).

4. SEMIGROUPS

In this section, we study so called *good* semigroup ideals defined by properties satisfied by value semigroup ideals (see Proposition 3.1.8 and Corollary 3.2.3).

4.1. Good semigroups and their ideals. Let S be a cancellative commutative monoid. Then S embeds into its (free abelian) group of differences D_S . If S is partially ordered, then D_S carries a natural induced partial order.

Definition 4.1.1. Let S be a partially ordered cancellative commutative monoid such that $\alpha \geq 0$ for all $\alpha \in S$. Assume that D_S is generated by a finite set I such that the isomorphism $D_S \cong \mathbb{Z}^I$ preserves the natural partial orders. Note that I is unique and contains only positive elements by Lemma 3.1.6. We set $\overline{S} := \{\alpha \in D_S \mid \alpha \geq 0\} \cong \mathbb{N}^I$.

We call S a *good semigroup* if properties (E0), (E1) and (E2) hold for $E = S$. If 0 is the only element of S with a zero component in D_S , then we call S *local*.

A *semigroup ideal* of a good semigroup S is a subset $\emptyset \neq E \subset D_S$ such that $E + S \subset E$. We always require that $\alpha + E \subset S$ for some $\alpha \in S$.

If E satisfies (E1), then we denote by $\mu^E := \min E$ its *minimum* which exists due to Dickson's lemma [Dic13].

If E satisfies (E1) and (E2), then we call E a *good semigroup ideal* of S . The set of good semigroup ideals of S is denoted by \mathfrak{G}_S .

Remark 4.1.2.

- (a) Any semigroup ideal E of S satisfies property (E0) because S does and $E + S \subset E$.
- (b) If $S \subset S' \subset \overline{S}$ are good semigroups, then $\mathfrak{G}_{S'} \subset \mathfrak{G}_S$. In particular, $S' \in \mathfrak{G}_S$.
- (c) Let R be an admissible ring. Then $S := \Gamma_R$ is a good semigroup, and $\Gamma_{\mathcal{E}} \in \mathfrak{G}_S$ for any $\mathcal{E} \in \mathfrak{R}_R$ (see Proposition 3.1.8.(a) and (b) and Corollary 3.2.3).
- (d) In general, Γ does not respect the multiplicative structure of \mathfrak{R}_R for admissible rings R , and \mathfrak{G}_S is not a monoid for good semigroups S (see Example 4.1.3 below).

Example 4.1.3. Consider the admissible ring

$$R := \mathbb{C}[(-t_1^4, t_2), (-t_1^3, 0), (0, t_2), (t_1^5, 0)] \subset \mathbb{C}[[t_1]] \times \mathbb{C}[[t_2]] = \overline{R}$$

with value semigroup $S := \Gamma_R$. Consider the R -submodules of Q_R

$$\mathcal{E} := \langle (t_1^3, t_2), (t_1^2, 0) \rangle_R, \quad \mathcal{F} := \langle (t_1^3, t_2), (t_1^4, 0), (t_1^5, 0) \rangle_R$$

with corresponding value semigroup ideals $E := \Gamma_{\mathcal{E}}$ and $F := \Gamma_{\mathcal{F}}$. Then $\mathcal{E}, \mathcal{F}, \mathcal{EF} \in \mathfrak{R}_R$, and hence $E, F, \Gamma_{\mathcal{EF}} \in \mathfrak{G}_S$ by Remark 4.1.2.(c). Figure 1 illustrates S, E, F and $E + F$. Obviously (E2) fails for $E + F$, and hence $E + F \notin \mathfrak{G}_S$. It follows that $\Gamma_{\mathcal{E}} + \Gamma_{\mathcal{F}} \subsetneq \Gamma_{\mathcal{EF}}$.

The following result reduces the study of good or value semigroups and their ideals to the local case.

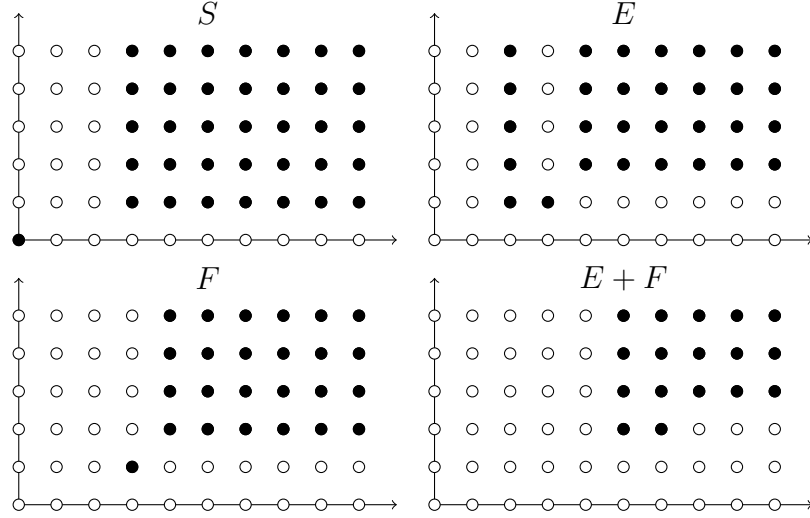


FIGURE 1. The value semigroup (ideals) in Example 4.1.3.

Theorem 4.1.4.

- (a) Let S be a good semigroup. Then there is a unique partition M of I and a unique isomorphism of partially ordered abelian groups $D_S \cong \prod_{m \in M} D_{S,m}$, where $D_{S,m} \cong \mathbb{Z}^m$ for any $m \in M$. Denoting by $\pi_m: D_S \rightarrow D_{S,m}$ the corresponding projections, the $S_m := \pi_m(S)$ are local good semigroups, and any semigroup ideal E of S satisfying property (E1) decomposes as

$$E = \prod_{m \in M} E_m,$$

where each $E_m := \pi_m(E)$ satisfies property (E1). Moreover, $E \in \mathfrak{G}_S$ implies $E_m \in \mathfrak{G}_{S_m}$.

- (b) Let R be an admissible ring. Then there is a bijection $\varphi: \text{Max}(R) \rightarrow M$ such that

$$(\Gamma_{\mathcal{E}})_{\varphi(\mathfrak{m})} = \Gamma_{\mathcal{E}_{\mathfrak{m}}}$$

for any $\mathcal{E} \in \mathfrak{R}_R$.

Proof. This follows from [BDF00, Thm. 2.5, Prop. 2.12], Lemma 3.1.6, Proposition 3.1.3, and Theorem 3.2.2. \square

For the remainder of this section, we identify $D_S \cong \mathbb{Z}^I$ as in Definition 4.1.1.

Definition 4.1.5. Let E and F be semigroup ideals of a good semigroup S . We write

$$E - F := \{\alpha \in D_S \mid \alpha + F \subset E\},$$

and we call

$$C_E := E - \overline{S} = \{\alpha \in D_S \mid \alpha + \overline{S} \subset E\}$$

the *conductor ideal* of E . We set $C := C_S$.

Remark 4.1.6. Let S be a good semigroup and $\alpha \in D_S$.

- (a) For any $E \in \mathfrak{G}_S$, $\alpha + E \in \mathfrak{G}_S$.
- (b) For any two semigroup ideals E and F of S , $(\alpha + E) - F = \alpha + (E - F)$ and $E - (\alpha + F) = -\alpha + (E - F)$.
- (c) For any two inclusions $E \subset E'$ and $F \subset F'$ of semigroup ideals of S , $E - F' \subset E - F \subset E' - F$.
- (d) For any $E \in \mathfrak{G}_S$, $E - S = E$.

Lemma 4.1.7. *For any two semigroup ideals E and F of S also $E - F$ is a semigroup ideal of S . If E satisfies (E1), so does $E - F$, and $C_E \in \mathfrak{G}_S \cap \mathfrak{G}_{\overline{S}}$.*

Proof. Since F is a semigroup ideal of S , we have $(E - F) + S + F = (E - F) + F \subset E$, and hence $(E - F) + S \subset E - F$. Since E is a semigroup ideal of S , there is $\alpha \in D_S$ such that $\alpha + E \subset S$. Then we have for any $\beta \in F$, $\alpha + \beta + (E - F) \subset \alpha + E \subset S$. Thus, $E - F$ is a semigroup ideal of S .

Assume now that E satisfies property (E1). Then for any $\alpha, \beta \in E - F$ and $\delta \in F$ we have $\min\{\alpha, \beta\} + \delta = \min\{\alpha + \delta, \beta + \delta\} \in E$ since $\alpha + \delta, \beta + \delta \in E$. Hence, $\min\{\alpha, \beta\} \in E - F$, and $E - F$ satisfies property (E1).

We have $C_E + \overline{S} + \overline{S} = (E - \overline{S}) + \overline{S} + \overline{S} = (E - \overline{S}) + \overline{S} \subset E$, and hence $C_E + \overline{S} \subset E - \overline{S} = C_E$. Therefore, C_E is a semigroup ideal of \overline{S} . As just shown it satisfies (E1), and hence $\min\{\alpha, \beta\} + \overline{S} \subset C_E$ for any $\alpha, \beta \in C_E$. It follows that C_E satisfies (E2). \square

Notation 4.1.8. Let S be a good semigroup, and let E be a semigroup ideal of S satisfying (E1). Then

$$\gamma^E := \mu^{C_E} = \min\{\alpha \in D_S \mid \alpha + \overline{S} \subset E\}.$$

is called the *conductor* of E . We abbreviate $\tau^E := \gamma^E - \mathbf{1}$, $\gamma := \gamma^S$ and $\tau := \tau^S$.

Remark 4.1.9. In general, $E - F$ does not satisfy (E2) for $E, F \in \mathfrak{G}_S$ (see [BDF00, Exa. 2.10]).

The following objects were introduced by Delgado [Del87, Del88] for investigating the Gorenstein symmetry. They measure jumps in the filtration \mathcal{Q}^α from the proof of Theorem 3.3.4 (see [CDK94, Rem. 4.6]).

Definition 4.1.10. Let S be a good semigroup, E is a semigroup ideal of S , $\alpha \in D_S$ and $J \subset I$. We define:

- (a) $\Delta_J(\alpha) := \{\beta \in D_S \mid \alpha_i = \beta_i \text{ for } i \in J \text{ and } \alpha_j < \beta_j \text{ for } j \notin J\}$
- (b) $\Delta_J^E(\alpha) := \Delta_J(\alpha) \cap E$
- (c) $\Delta(\alpha) := \bigcup_{i \in I} \Delta_i(\alpha)$, where $\Delta_i(\alpha) := \Delta_{\{i\}}(\alpha)$
- (d) $\Delta^E(\alpha) := \Delta(\alpha) \cap E$

In the remainder of this subsection, we provide some technical preliminaries which will be used in §5. The statements of the following two lemmas were proved in [Del88, Lem. 1.8 and Cor. 1.9] in case $E = S$.

Lemma 4.1.11. *Let S be a good semigroup and $E \in \mathfrak{G}_S$. Assume that there is $\alpha \in E$ and $J \subset I$ such that $\alpha_j \geq \gamma_j^E$ for all $j \in J$. If $\delta \in D_S$ with*

$$\begin{aligned} \delta_j &\geq \gamma_j^E \text{ for all } j \in J, \\ \delta_k &= \alpha_k \text{ for all } k \in I \setminus J, \end{aligned}$$

then $\delta \in E$.

Proof. Choose $\beta \in D_S$ such that

$$\begin{aligned} \beta_j &= \alpha_j \text{ for all } j \in J, \\ \beta_i &> \max\{\gamma_i^E, \alpha_i\} \text{ for all } i \in I \setminus J. \end{aligned}$$

In particular, $\beta \geq \gamma^E$, and hence $\beta \in E$. Now applying property (E2) to α and β we obtain for any $j \in J$ an $\alpha' \in E$ with $\alpha' \geq \alpha + \mathbf{e}_j$. Therefore, we may assume $\alpha \geq \delta$.

Pick $\epsilon \in D_S$ such that

$$\begin{aligned}\epsilon_j &= \delta_j \text{ for all } j \in J, \\ \epsilon_i &> \max\{\gamma_i^E, \delta_i\} \text{ for all } i \in I \setminus J.\end{aligned}$$

In particular, $\epsilon \geq \gamma^E$, and hence $\epsilon \in E$. Thus, $\delta = \min\{\epsilon, \alpha\} \in E$ since E satisfies (E1). \square

Lemma 4.1.12. *Let S be a good semigroup. Then $\Delta^E(\tau^E) = \emptyset$ for any $E \in \mathfrak{G}_S$.*

Proof. Assume that $\Delta^E(\tau^E) \neq \emptyset$. Then there is $i \in I$ and $\beta \in \Delta_i^E(\tau^E)$. This yields

$$\gamma' = \min\{\gamma^E, \beta\} = \gamma^E - \mathbf{e}_i \in E$$

since E satisfies condition (E1). Thus, Lemma 4.1.11 implies $\gamma' + \overline{S} \subset E$, and hence $\gamma^E > \gamma' \in C_E$ contradicting the minimality of γ^E in C_E . \square

Lemma 4.1.13. *Let E and F be semigroup ideals of a good semigroup S satisfying property (E1). Then $\gamma^{E-F} = \gamma^E - \mu^F$.*

Proof. Note that γ^{E-F} is defined since $E - F$ satisfies property (E1) by Lemma 4.1.7. Since $F - \mu^F \subset \overline{S}$ and $\gamma^E + \overline{S} \subset E$, we have $\gamma^E - \mu^F + \overline{S} + F \subset \gamma^E + \overline{S} \subset E$, and hence $\gamma^E - \mu^F \geq \gamma^{E-F}$.

Conversely, $\gamma^{E-F} + \mu^F + \overline{S} = \gamma^{E-F} + \mu^F - \mu^F + F + \overline{S} = \gamma^{E-F} + F + \overline{S} \subset E$ implies $\gamma^{E-F} + \mu^F \geq \gamma^E$. \square

4.2. Length and distance.

Definition 4.2.1. Let S be a good semigroup, and let $E \subset D_S$ be a subset. Then $\alpha, \beta \in E$ with $\alpha < \beta$ are called *consecutive* in E if $\alpha < \delta < \beta$ implies $\delta \notin E$ for any $\delta \in D_S$. For $\alpha, \beta \in E$, a chain

$$(4.1) \quad \alpha = \alpha^{(0)} < \dots < \alpha^{(n)} = \beta$$

of points $\alpha^{(i)} \in E$ is said to be *saturated of length n* if $\alpha^{(i)}$ and $\alpha^{(i+1)}$ are consecutive in E for all $i \in \{0, \dots, n-1\}$. If E satisfies

(E4) For fixed $\alpha, \beta \in E$, any two saturated chains (4.1) in E have the same length n .
then we call $d_E(\alpha, \beta) := n$ the *distance* of α and β in E .

Proposition 4.2.2. *Let S be a good semigroup. Then any $E \in \mathfrak{G}_S$ satisfies property (E4).*

Proof. See [D'A97, Prop. 2.3]. \square

Definition 4.2.3. Let S be a good semigroup, and let $E \subset F$ be two semigroup ideals of S satisfying property (E4). Then we call

$$d(F \setminus E) := d_F(\mu^F, \gamma^E) - d_E(\mu^E, \gamma^E)$$

the *distance* between E and F .

Remark 4.2.4. Let $E \subset F$ be two semigroup ideals of a good semigroup S satisfying property (E4).

- (a) If E and F satisfy property (E1), then $d(F \setminus E) = \sum_{m \in M} d(F_m \setminus E_m)$ using the notation from Theorem 4.1.4 (see [BDF00, Prop. 2.12]).

(b) If $\epsilon \geq \gamma^E$, then

$$\begin{aligned} d(F \setminus E) &= d_F(\mu^F, \gamma^E) - d_E(\mu^E, \gamma^E) \\ &= d_F(\mu^F, \gamma^E) + d_F(\gamma^E, \epsilon) - d_E(\mu^E, \gamma^E) - d_E(\gamma^E, \epsilon) \\ &= d_F(\mu^F, \epsilon) - d_E(\mu^E, \epsilon) \end{aligned}$$

by additivity of $d(-, -)$ and since $d_F(\gamma^E, \epsilon) = d_E(\gamma^E, \epsilon)$.

In the following, we collect the main properties of the distance function $d(- \setminus -)$. It is obvious from the definition that it is additive.

Lemma 4.2.5. *Let $E \subset F \subset G$ be semigroup ideals of a good semigroup S satisfying property (E4). Then*

$$d(G \setminus E) = d(G \setminus F) + d(F \setminus E).$$

Proof. See [D'A97, Prop. 2.7]. □

The distance function detects equality as formulated in [D'A97, Prop. 2.8].

Proposition 4.2.6. *Let S be a good semigroup, and let $E, F \in \mathfrak{G}_S$ with $E \subset F$. Then $E = F$ if and only if $d(F \setminus E) = 0$.*

The proof of Proposition 4.2.6 is an immediate consequence of the following lemma.

Lemma 4.2.7. *Let $E \subset F$ be two semigroup ideals of a good semigroup S , where $E \in \mathfrak{G}_S$ and F satisfies property (E1). Let $\alpha \in F \setminus E$ be minimal. Then any $\beta \in E$ maximal with $\beta < \alpha$ and $\beta' \in E$ minimal with $\alpha < \beta'$ are consecutive in E .*

Proof. Suppose that there exists an $\epsilon \in E$ such that $\beta < \epsilon < \beta'$. By choice of β and β' , $\alpha \not\leq \epsilon \not\leq \alpha$, and hence $\min\{\alpha, \epsilon\} < \alpha$. Since F satisfies property (E1), $\min\{\alpha, \epsilon\} \in F$. Hence, $\min\{\alpha, \epsilon\} \in E$ by minimality of $\alpha \in F \setminus E$, and $\min\{\alpha, \epsilon\} = \beta$ by maximality of β . Then $\min\{\beta, \epsilon\} = \beta$, and there are $j, k \in I$ such that $\beta_j = \epsilon_j$ and $\beta_k \neq \epsilon_k$. Applying property (E2) to $\beta, \epsilon \in E$ yields an $\epsilon' \in E$ such that

$$\begin{aligned} \epsilon'_j &> \beta_j = \epsilon_j < \alpha_j \leq \beta'_j, \\ \epsilon'_i &\geq \beta_i \text{ for all } i \in I, \\ \epsilon'_k &= \beta_k < \epsilon_k \leq \beta'_k. \end{aligned}$$

Since E satisfies property (E1), we may replace ϵ' by $\min\{\epsilon', \beta'\} \in E$ and assume that $\beta < \epsilon' < \beta'$. Then the choices of β and β' imply $\alpha \not\leq \epsilon' \not\leq \alpha$. Therefore, $\alpha' := \min\{\epsilon', \alpha\} \in F$, and $\beta < \alpha' < \alpha$. Now both possibilities $\alpha' \notin E$ and $\alpha' \in E$ yield a contradiction, either to the minimality of $\alpha \in F \setminus E$ or to the maximality of $\beta \in E$ with $\beta < \alpha$. □

Proof of Proposition 4.2.6. For the non-trivial implication, assume that $d(F \setminus E) = 0$ but $E \subsetneq F$. Pick $\alpha \in F \setminus E$ minimal. In particular, $\mu^E < \alpha < \gamma^E$. By Lemma 4.2.7 there are $\beta, \beta' \in E$ which are consecutive in E but not in F . Since E satisfies property (E4) (see Proposition 4.2.2), a saturated chain

$$\mu^E = \beta^{(0)} < \dots < \beta^{(i)} = \beta < \beta' = \beta^{(i+1)} < \dots < \beta^{(n)} = \gamma^E$$

in E has length $n = d_E(\mu^E, \gamma^E)$. Extending it to a chain

$$\mu^F \leq \beta^{(0)} < \dots < \beta^{(i)} = \beta < \alpha < \beta' = \beta^{(i+1)} < \dots < \beta^{(n)} = \gamma^E$$

in F by inserting α yields $d_F(\mu^F, \gamma^E) > n = d_E(\mu^E, \gamma^E)$, contradicting the hypothesis. □

Finally, we show that the distance function coincides with the relative length of fractional ideals when evaluated on their value semigroup ideals.

Proposition 4.2.8. *Let R be an admissible ring. If $\mathcal{E}, \mathcal{F} \in \mathfrak{R}_R$ such that $\mathcal{E} \subset \mathcal{F}$, then*

$$\ell_R(\mathcal{F}/\mathcal{E}) = d(\Gamma_{\mathcal{F}} \setminus \Gamma_{\mathcal{E}}).$$

Proof. See [D'A97, Prop. 2.2] for part of the following proof in the local case. By Proposition 3.1.8, $E := \Gamma_{\mathcal{E}}$ and $F := \Gamma_{\mathcal{F}}$ are good semigroup ideals of Γ_R and hence satisfy property (E4) (see Proposition 4.2.2).

Let \mathfrak{j} be the Jacobson radical of R . Then $\mathfrak{j} \subset \mathfrak{n}$ for all $\mathfrak{n} \in \text{Max } \overline{R}$. By Theorem 3.1.1.(d), $\nu(xy) \geq \mu_F + n \cdot (1, \dots, 1)$ for any $x \in \mathfrak{j}^n$ and $y \in \mathcal{F}$. By Lemma 3.1.7 and Notation 3.3.3, $\mathcal{C}_{\mathcal{E}} = Q^{\alpha}$ for some $\alpha \in \mathbb{Z}^{\mathfrak{B}_R}$. It follows that $\mathfrak{j}^n \mathcal{F} \subset Q^{\mu_F + n \cdot (1, \dots, 1)} \subset \mathcal{C}_{\mathcal{E}} \subset \mathcal{E}$ for large enough n . This turns \mathcal{F}/\mathcal{E} into an R/\mathfrak{j}^n -module. Since $R/\mathfrak{j}^n = \prod_{\mathfrak{m} \in \text{Max}(R)} (R/\mathfrak{j}^n)_{\mathfrak{m}}$ (see [Nag62, Proof of Thm. 17.7]), $\mathcal{F}/\mathcal{E} = \prod_{\mathfrak{m} \in \text{Max}(R)} \mathcal{F}_{\mathfrak{m}}/\mathcal{E}_{\mathfrak{m}}$, and hence

$$\ell_R(\mathcal{F}/\mathcal{E}) = \sum_{\mathfrak{m} \in \text{Max}(R)} \ell_{R_{\mathfrak{m}}}(\mathcal{F}_{\mathfrak{m}}/\mathcal{E}_{\mathfrak{m}}).$$

Due to Theorem 4.1.4.(b) and Remark 4.2.4.(a) we may assume now that R is local.

Since R is residually rational, for all $V \in \mathfrak{V}_R$,

$$\mathcal{E}^{\alpha}/\mathcal{E}^{\alpha+e_V} \subset Q^{\alpha}/Q^{\alpha+e_V} \xrightarrow[\cong]{\cdot \frac{1}{x}} V/\mathfrak{m}_V = R/\mathfrak{m},$$

where $x \in Q^{\text{reg}}$ with $\nu(x) = \alpha$ (see Notation 3.3.3 and the proof of Proposition 3.1.8.(d)). Thus, $\ell_R(\mathcal{E}^{\alpha}/\mathcal{E}^{\alpha+e_V}) \leq 1$, and $\alpha \in E$ if and only if $\ell_R(\mathcal{E}^{\alpha}/\mathcal{E}^{\alpha+e_V}) = 1$ for all $V \in \mathfrak{V}_R$. If α and β are consecutive points in E , then $d_E(\alpha, \beta) = 1$ by definition. We claim that $\ell_R(\mathcal{E}^{\alpha}/\mathcal{E}^{\beta}) = 1$. By additivity of the length, it suffices to show that for any $\alpha < \delta < \beta$, $\ell_R(\mathcal{E}^{\delta}/\mathcal{E}^{\delta+e_W}) = 0$ with $\delta+e_W \leq \beta$ for a suitable $W \in \mathfrak{V}_R$. Since $\delta \notin E$, there is a $W \in \mathfrak{V}_R$ such that $\ell_R(\mathcal{E}^{\delta}/\mathcal{E}^{\delta+e_W}) = 0$. Moreover, if $\delta_W = \beta_W$, then $\mathcal{E}^{\beta}/\mathcal{E}^{\beta+e_W} \subset \mathcal{E}^{\delta}/\mathcal{E}^{\delta+e_W}$ since $\delta < \beta$. Hence, $\ell_R(\mathcal{E}^{\delta}/\mathcal{E}^{\delta+e_W}) \geq \ell_R(\mathcal{E}^{\beta}/\mathcal{E}^{\beta+e_W}) = 1$ since $\beta \in E$. It follows that $\delta_W < \beta_W$ as desired.

Pick $\epsilon \in \Gamma_{\mathcal{E}} \subset C_E$, hence $\epsilon \geq \gamma^E$. Then the additivity of the distance yields that

$$d_E(\mu^E, \epsilon) = \ell_R(\mathcal{E}^{\mu^E}, \mathcal{E}^{\epsilon}) = \ell_R(\mathcal{E}, \mathcal{E}^{\epsilon}).$$

As the above arguments are valid also for F and \mathcal{F} in place of E and \mathcal{E} respectively, we conclude, using Remark 4.2.4.(b), that

$$\begin{aligned} d(F \setminus E) &= d_F(\mu^F, \epsilon) - d_E(\mu^E, \epsilon) \\ &= \ell_R(\mathcal{F}/\mathcal{F}^{\epsilon}) - \ell_R(\mathcal{E}/\mathcal{E}^{\epsilon}) = \ell_R(\mathcal{F}/\mathcal{E}^{\epsilon}) - \ell_R(\mathcal{E}/\mathcal{E}^{\epsilon}) = \ell_R(\mathcal{F}/\mathcal{E}). \end{aligned} \quad \square$$

Corollary 4.2.9. *Let R be an admissible ring, and let $\mathcal{E}, \mathcal{F} \in \mathfrak{R}_R$ such that $\mathcal{E} \subset \mathcal{F}$. Then $\mathcal{E} = \mathcal{F}$ if and only if $\Gamma_{\mathcal{E}} = \Gamma_{\mathcal{F}}$.*

Proof. This follows immediately from Remark 4.1.2.(c) and Propositions 4.2.6 and 4.2.8 (see [D'A97, Cor. 2.5]). \square

5. DUALITY

Let R be an admissible local ring with value semigroup $S := \Gamma_R$. Generalizing earlier results of Jäger [Jäg77] and Delgado [Del87, Del88], D'Anna [D'A97] characterized (suitably normalized) canonical fractional ideals \mathcal{K} of R by having a certain value semigroup ideal $\Gamma_{\mathcal{K}} = K_S^0$, defined purely and explicitly in semigroup terms. Based on this result, we define and study canonical (semigroup) ideals of good semigroups in analogy with the ring case. We give three equivalent definitions of canonical ideals, show that dualizing with them preserves the property of being a good semigroup ideal, and describe the behaviour under extension of the semigroup. Finally, we show that value semigroup ideals

are compatible with dualizing in the sense that for canonical ideals \mathcal{K} and $K = \Gamma_{\mathcal{K}}$ the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{R}_R & \xrightarrow{\mathcal{E} \mapsto \mathcal{K} : \mathcal{E}} & \mathfrak{R}_R \\ \mathcal{E} \mapsto \Gamma_{\mathcal{E}} \downarrow & & \downarrow \mathcal{E} \mapsto \Gamma_{\mathcal{E}} \\ \mathfrak{G}_{\Gamma_R} & \xrightarrow{E \mapsto K - E} & \mathfrak{G}_{\Gamma_R}. \end{array}$$

5.1. Canonical ideals. We recall the well-known basic facts about existence and uniqueness of canonical ideals of R .

Definition 5.1.1. Let R be a one-dimensional Cohen–Macaulay ring. A regular fractional ideal $\mathcal{K} \in \mathfrak{R}_R$ is said to be a *canonical (fractional) ideal* of R if

$$(5.1) \quad \mathcal{E} = \mathcal{K} : (\mathcal{K} : \mathcal{E})$$

for all $\mathcal{E} \in \mathfrak{R}_R$. If R is a canonical ideal, then R is called a *Gorenstein ring*.

In other words, for $\mathcal{F} = \mathcal{K}$ all the maps in Diagram (2.2) are isomorphisms. In particular, this means that

$$(5.2) \quad \mathcal{K} : \mathcal{K} = R.$$

By Lemma 2.1.4.(b), Diagram (2.2) and the compatibility of Hom with flat base change, R has a canonical ideal \mathcal{K} if and only if its completion \widehat{R} has a canonical ideal $\widehat{\mathcal{K}}$ (see [HK71, Lem. 2.10]).

Remark 5.1.2. Let R be a one-dimensional Cohen–Macaulay ring. It is not difficult to see that a canonical ideal \mathcal{K} of R is a faithful maximal Cohen–Macaulay module of R . Moreover, since dualizing with \mathcal{K} preserves the length (see [HK71, Rem. 2.5.(c)]), it follows that \mathcal{K} is of type one. Therefore, canonical ideals are canonical modules (see [BH93, Prop. 3.3.13 and Def. 3.3.16]).

Concerning uniqueness of canonical ideals the following is known.

Proposition 5.1.3. *Let R be a one-dimensional Cohen–Macaulay ring with a canonical ideal \mathcal{K} . Then \mathcal{K}' is a canonical ideal of R if and only if $\mathcal{K}' = \mathcal{E}\mathcal{K}$ for some invertible ideal \mathcal{E} of R . In case R is local, the latter condition becomes $\mathcal{K}' = a\mathcal{K}$ for some $a \in Q_R^{\text{reg}}$ (see §2.1).*

Proof. See [HK71, Satz 2.8]. □

In case R is local, the existence of a canonical ideal of R can be characterized as follows.

Theorem 5.1.4. *A one-dimensional local Cohen–Macaulay ring R has a canonical ideal if and only if \widehat{R} is generically Gorenstein. In particular, any one-dimensional analytically reduced local Cohen–Macaulay ring has a canonical ideal.*

Proof. See [HK71, Kor. 2.12, Satz 6.21]. □

Corollary 5.1.5. *Any one-dimensional analytically reduced local Cohen–Macaulay ring R with large residue field has a unique canonical ideal \mathcal{K}_R^0 such that*

$$R \subset \mathcal{K}_R^0 \subset \overline{R}.$$

If \mathcal{K}' is another canonical ideal of R , then $\mathcal{K}' = a\mathcal{K}_R^0$ for some $a \in Q_R^{\text{reg}}$.

Proof. Applying Lemma 2.1.5, Theorem 3.1.1.(c) and (d), Proposition 5.1.3 and Theorem 5.1.4 to $S = \overline{R}$ yields the statement. □

Lemma 5.1.6. *Let $\varphi: R \rightarrow R'$ be a local homomorphism of one-dimensional local Cohen–Macaulay rings such that $Q_R = Q_{R'}$. If \mathcal{K}_R is a canonical ideal of R , then*

$$\mathcal{K}_{R'} \cong \mathcal{K}_R : R'.$$

Proof. This follows immediately from (2.3) and (5.1). \square

5.2. Duality on good semigroups. The following Theorem 5.2.4 due to D’Anna characterizes the canonical ideals by having a certain value semigroup ideal defined as follows.

Definition 5.2.1. For any good semigroup S , we call

$$K_S^0 := \{\alpha \in D_S \mid \Delta^S(\tau - \alpha) = \emptyset\}.$$

the (normalized) canonical (semigroup) ideal of S .

Lemma 5.2.2. *Let S be a good semigroup. Then the set K_S^0 is a semigroup ideal of S satisfying property (E1) with minimum $\mu^{K_S^0} = \mu^S = 0$ and conductor $\gamma^{K_S^0} = \gamma$.*

Proof. See [D’A97, Prop. 3.2], Lemma 4.1.12 and Notation 4.1.8. \square

The following proposition was stated in [BDF00, Prop. 2.15].

Proposition 5.2.3. *Let $S = \prod_{m \in M} S_m$ be the decomposition of the good semigroup S into local good semigroups S_m (see Theorem 4.1.4). Then*

$$K_S^0 = \prod_{m \in M} K_{S_m}^0. \quad \square$$

Theorem 5.2.4. *Let R be an admissible local ring with value semigroup $S := \Gamma_R$. Then for any fractional ideal \mathcal{K} of R such that $R \subset \mathcal{K} \subset \overline{R}$ the following are equivalent:*

- (i) \mathcal{K} is a canonical ideal of R .
- (ii) $\mathcal{K} = \mathcal{K}_R^0$ (see Corollary 5.1.5)
- (iii) $\Gamma_{\mathcal{K}} = K_S^0$ (see Definition 5.2.1)

In particular, $\Gamma_{\mathcal{K}_R^0} = K_S^0$.

Proof. See [D’A97, Thm. 4.1]. \square

We shall generalize Theorem 5.2.4 to the semilocal case in Corollary 5.3.6. Our definition of a canonical semigroup ideal below allows for shifts.

Definition 5.2.5. Let S be a good semigroup (see Definition 4.1.1). We call $K \in \mathfrak{G}_S$ a canonical (semigroup) ideal of S if $K \subset E$ implies $K = E$ for all $E \in \mathfrak{G}_S$ with $\gamma^K = \gamma^E$.

Remark 5.2.6. If K is a canonical ideal of S , then $\alpha + K$ is a canonical ideal of S for all $\alpha \in D_S$. In fact, this follows immediately from Definition 5.2.5 and Remark 4.1.6.(a).

Our aim in this section is to establish the following result on canonical semigroup ideals in analogy with the ring case.

Theorem 5.2.7. *Let S be a good semigroup. For any $K \in \mathfrak{G}_S$, the following are equivalent:*

- (i) K is a canonical ideal of S .
- (ii) There is $\alpha \in D_S$ such that $\alpha + K = K_S^0$.
- (iii) For all $E \in \mathfrak{G}_S$ we have $K - (K - E) = E$.

If K is a canonical ideal of S , then the following hold:

- (a) If $S \subset K \subset \overline{S}$, then $K = K_S^0$.
- (b) If $E \in \mathfrak{G}_S$, then $K - E \in \mathfrak{G}_S$.

(c) $K - K = S$.

(d) If $S' \subset \overline{S}$ is a good semigroup with $S \subset S'$, then $K' = K - S'$ is a canonical ideal of S' .

Proof.

(i) \Rightarrow (ii) See Proposition 5.2.11.

(ii) \Rightarrow (iii) See Remark 4.1.6.(b) and Proposition 5.2.16.

(iii) \Rightarrow (i) See Proposition 5.2.14.

(a) See Lemma 5.2.2 and Proposition 5.2.11.

(b) See Remark 4.1.6.(a) and (b) and Propositions 5.2.10 and 5.2.11.

(c) See Remark 4.1.6.(b) and (d) and Propositions 5.2.11 and 5.2.16.

(d) See Corollary 5.2.12. □

Remark 5.2.8. The assumption $E \in \mathfrak{G}_S$ in Theorem 5.2.7.(iii) and (b) is necessary (see the example given by Figure 2).

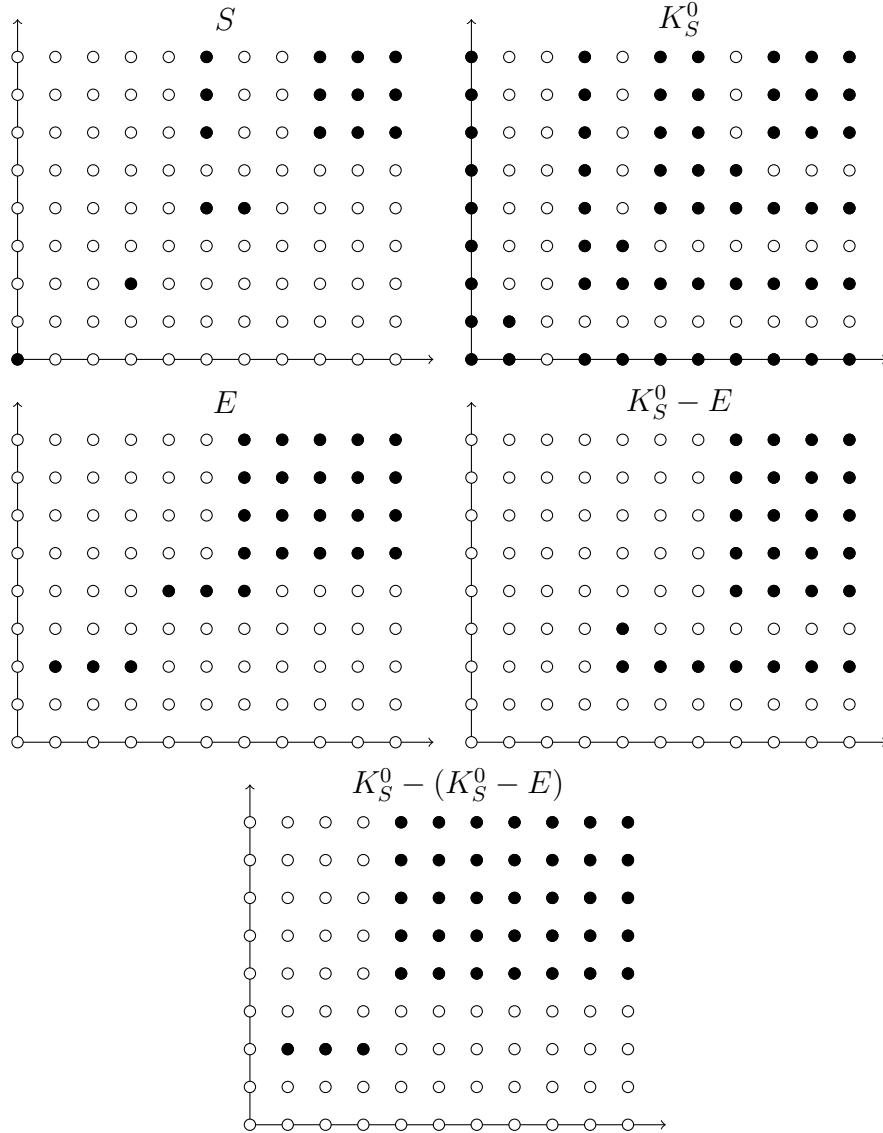


FIGURE 2. A semigroup ideal E satisfying property (E1) but not (E2), where $K_S^0 - E \notin \mathfrak{G}_S$ and $E \subsetneq K_S^0 - (K_S^0 - E)$.

We first approach Part (b) of Theorem 5.2.7 in case $K = K_S^0$. To this end we collect some properties of K_S^0 .

Lemma 5.2.9. *Let S be a good semigroup. Then the semigroup ideal K_S^0 of S has the following properties:*

- (a) $\Delta^{K_S^0}(\tau) = \emptyset$.
- (b) If E is a semigroup ideal of S , then $K_S^0 - E = \{\alpha \in D_S \mid \Delta^E(\tau - \alpha) = \emptyset\}$.

Proof. This follows immediately from Definitions 4.1.10 and 5.2.1 (see [D'A97, Comp. 3.3, Lem. 3.4]). \square

The proof of Theorem 5.2.7.(b) in case $K = K_S^0$ is achieved by the following proposition.

Proposition 5.2.10. *Let S be a good semigroup. Then $K_S^0 - E \in \mathfrak{G}_S$ for any $E \in \mathfrak{G}_S$. In particular, $K_S^0 \in \mathfrak{G}_S$.*

Proof. The idea of the following proof is illustrated in Figure 3.

Suppose that $K_S^0 - E \notin \mathfrak{G}_S$. Since $K_S^0 - E$ is a semigroup ideal of S satisfying property (E1) by Lemmas 4.1.7 and 5.2.2, it violates property (E2). That is, there are $\alpha, \beta \in K_S^0 - E$ with $\emptyset \neq J := \{j \in I \mid \alpha_j \neq \beta_j\} \subset I$, $\zeta^{(0)} := \min\{\alpha, \beta\} \in K_S^0 - E$, and $l_0 \in I \setminus J$ such that $\zeta \notin K_S^0 - E$ whenever $\zeta_{l_0} > \zeta_{l_0}^{(0)}$, $\zeta_i \geq \zeta_i^{(0)}$ for all $i \in I$, and $\zeta_j = \zeta_j^{(0)}$ for all $j \in J$. In particular, any choice of $l_1, l_2, \dots \in I \setminus J$ yields

$$\zeta^{(0)} \in K_S^0 - E, \quad \zeta^{(r)} := \zeta^{(r-1)} + \mathbf{e}_{l_{r-1}} \notin K_S^0 - E.$$

By Lemma 5.2.9.(b) this means that $\Delta^E(\tau - \zeta^{(0)}) = \emptyset$, and there is a $k_r \in I$ such that there is a $\delta^{(r)} \in \Delta_{k_r}^E(\tau - \zeta^{(r)}) \neq \emptyset$ for all $r \geq 1$. In order to construct a sequence as above, we proceed by induction on r . In each step we show that $\Delta_j^E(\tau - \zeta^{(r)}) = \emptyset$ for all $j \in J$, and we choose $l_r := k_r$.

Assume this was done for $r - 1$, and suppose $k_r \in J$. In particular, $k_r \neq l_{r-1}$. By definition of $\zeta^{(r)}$, this implies that for $r \geq 1$ there is a

$$(5.3) \quad \delta^{(r)} \in \Delta_{k_r}^E(\tau - \zeta^{(r)}) = \Delta_{k_r}^E(\tau - \zeta^{(r-1)}) \sqcup \Delta_{\{k_r, l_{r-1}\}}^E(\tau - \zeta^{(r-1)}) = \Delta_{\{k_r, l_{r-1}\}}^E(\tau - \zeta^{(r-1)}),$$

where the last equality follows from the inductive hypothesis. We deduce a contradiction with different arguments for $r = 1$ and $r \geq 2$, respectively.

First consider the case $r = 1$. Since $\beta \in K_S^0 - E$ and $\delta^{(1)} \in E$, we get $\delta^{(1)} + \beta \in K_S^0$. As $k_1 \in J$, we may assume $\beta_{k_1} > \alpha_{k_1} = \zeta_{k_1}^{(0)}$. By (5.3), $\delta^{(1)} + \zeta^{(0)} \in \Delta_{\{k_1, l_0\}}^E(\tau)$, and this implies $\delta^{(1)} + \beta \in \Delta_{l_0}^{K_S^0}(\tau)$, contradicting Lemma 5.2.9.(a).

Assume now that $r \geq 2$. By (5.3), $\delta_{l_{r-1}}^{(r)} = \delta_{l_{r-1}}^{(r-1)}$. Then property (E2) applied to $\delta^{(r-1)}, \delta^{(r)} \in E$ yields $\varepsilon \in E$ with $\varepsilon_{l_{r-1}} > \delta_{l_{r-1}}^{(r)}$, $\varepsilon_i \geq \min\{\delta_i^{(r-1)}, \delta_i^{(r)}\}$ for all $i \in I$ and equality if $\delta_i^{(r-1)} \neq \delta_i^{(r)}$. Again by (5.3) and since $k_r \neq l_{r-1}$, $\delta_{k_r}^{(r-1)} > \tau_{k_r} - \zeta_{k_r}^{(r-1)} = \delta_{k_r}^{(r)} = \varepsilon_{k_r}$. It follows that $\varepsilon \in \Delta_{k_r}^E(\tau - \zeta^{(r-1)})$, contradicting the induction hypothesis.

Now pick $r > \sum_{k \in I \setminus J} |\tau_k - \zeta_k^{(1)} - \mu_k^E|$. Then $\delta_{l_r}^{(r)} = \tau_{l_r} - \zeta_{l_r}^{(r)} < \mu_{l_r}^E$, contradicting the minimality of μ^E . \square

We can now relate our canonical ideals (see Definition 5.2.5) to D'Anna's normalized one (see Definition 5.2.1).

Proposition 5.2.11. *Let S be a good semigroup, and let $K \in \mathfrak{G}_S$. Then K is a canonical ideal of S if and only if $K = \alpha + K_S^0$ for some $\alpha \in D_S$. In particular, for any $\delta \in D_S$ there is a unique canonical ideal K of S with $\gamma^K = \delta$.*

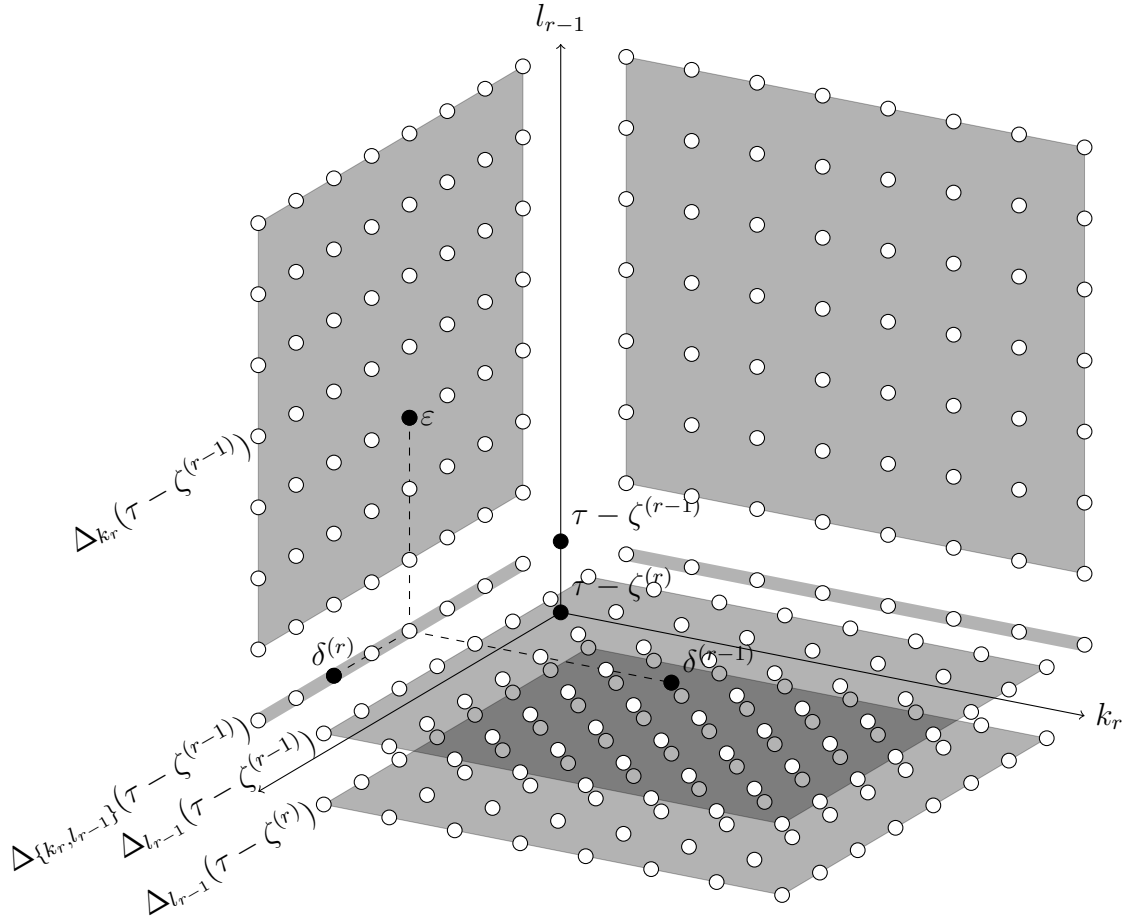


FIGURE 3. Induction step in the proof of Proposition 5.2.10 in case $I \setminus J = \{l_{r-1}\}$.

Proof. Using Remark 5.2.6, it suffices to show that K_S^0 is the unique canonical ideal of S with conductor $\gamma^{K_S^0} = \gamma$ (see Lemma 5.2.2). To this end, let $E \in \mathfrak{G}_S$ with $\gamma^E = \gamma$, and assume there is a $\beta \in E \setminus K_S^0$. Then there is $\delta \in \Delta^S(\tau - \beta)$ (see Definition 5.2.1), and hence $\beta + \delta \in \Delta^E(\tau)$. This contradicts Lemma 4.1.12, and therefore $E \subset K_S^0$. Since $K_S^0 \in \mathfrak{G}_S$ by Proposition 5.2.10, it is a canonical ideal of S .

If K is another canonical ideal of S with conductor $\gamma^K = \gamma$, then $K \subset K_S^0$ (see above) yields $K = K_S^0$ (see Definition 5.2.5). \square

As a consequence we deduce the counterpart of the push-forward formula for canonical ideals (see Lemma 5.1.6) on the semigroup side (see Theorem 5.2.7.(d)).

Corollary 5.2.12. *Let $S \subset S' \subset \overline{S}$ be good semigroups. If K is a canonical ideal of S , then $K' = K - S'$ is a canonical ideal of S' .*

Proof. By Remark 4.1.2.(b), $S' \in \mathfrak{G}_S$, and, by Proposition 5.2.11, $K = \alpha + K_S^0$ for some $\alpha \in D_S$. Then, by Lemma 5.2.9.(b),

$$\begin{aligned} K' &= (\alpha + K_S^0) - S' \\ &= \alpha + (K_S^0 - S') \\ &= \alpha + \{\beta \in D_S \mid \Delta^{S'}(\tau - \beta) = \emptyset\} \\ &= \alpha + \tau - \tau^{S'} + \{\delta \in D_S \mid \Delta^{S'}(\tau^{S'} - \delta) = \emptyset\}. \end{aligned}$$

Thus, K' is a canonical ideal of S' by Proposition 5.2.11. \square

By the following two propositions we establish an equivalent definition of canonical semigroup ideals (see Theorem 5.2.7.(iii)) analogous to that of canonical fractional ideals (see Definition 5.1.1).

Lemma 5.2.13. *Let E and F be semigroup ideals of a good semigroup S .*

- (a) $E \subset F - (F - E)$.
- (b) If E and F satisfy property (E1), $F \subsetneq E$, and $\gamma^E = \gamma^F$, then $E \subsetneq F - (F - E)$.

Proof. (a) This follows trivially from Definition 4.1.5.

(b) By Lemma 4.1.13 and since $F \subsetneq E$ forces $\mu^{F-E} > 0$, we have

$$\gamma^{F-(F-E)} = \gamma^F - \mu^{F-E} < \gamma^F = \gamma^E.$$

Then the claim follows from (a). \square

Proposition 5.2.14. *Let S be a good semigroup, and let $K \in \mathfrak{G}_S$ such that $K - (K - E) = E$ for all $E \in \mathfrak{G}_S$. Then K is a canonical ideal of S .*

Proof. Assume that K is not a canonical ideal of S . By Proposition 5.2.11 there is a canonical ideal E of S with $\gamma^E = \gamma^K$, and hence $K \subsetneq E$ (see Definition 5.2.5). By Lemma 5.2.13 and the hypothesis this leads to the contradiction $E \subsetneq K - (K - E) = E$. \square

Lemma 5.2.15. *Let E be a semigroup ideal of a good semigroup S , and let $\alpha \in K_S^0 - (K_S^0 - E)$. If $\zeta \in D_S$ satisfies $\Delta^E(\tau - \zeta) = \emptyset$, then $\Delta^S(\tau - \zeta - \alpha) = \emptyset$. Equivalently, if $\beta \in D_S$ satisfies $\Delta^S(\tau - \beta) \neq \emptyset$, then $\Delta^E(\tau - \beta + \alpha) \neq \emptyset$.*

Proof. Using Lemma 5.2.9.(b) we have

$$\begin{aligned} K_S^0 - (K_S^0 - E) &= \{\alpha \in D_S \mid \alpha + (K_S^0 - E) \subset K_S^0\} \\ &= \{\alpha \in D_S \mid \alpha + \{\zeta \in D_S \mid \Delta^E(\tau - \zeta) = \emptyset\} \subset K_S^0\} \\ &= \{\alpha \in D_S \mid \Delta^S(\tau - \zeta - \alpha) = \emptyset \text{ for all } \zeta \in D_S \text{ such that } \Delta^E(\tau - \zeta) = \emptyset\}. \end{aligned}$$

Thus, if $\zeta \in D_S$ satisfies $\Delta^E(\tau - \zeta) = \emptyset$, then $\Delta^S(\tau - \zeta - \alpha) = \emptyset$ for all $\alpha \in K_S^0 - (K_S^0 - E)$. The equivalent claim follows by setting $\zeta = \beta - \alpha \in D_S$. \square

Proposition 5.2.16. *Let S be a good semigroup. Then $K_S^0 - (K_S^0 - E) = E$ for any $E \in \mathfrak{G}_S$.*

Proof. Note that the inclusion $E \subset K_S^0 - (K_S^0 - E)$ holds trivially by Lemma 5.2.13.(a). So assume that $E \subsetneq K_S^0 - (K_S^0 - E)$. By Lemma 4.1.7, $K_S^0 - (K_S^0 - E)$ is a semigroup ideal of S . In particular, there is a minimal $\alpha \in (K_S^0 - (K_S^0 - E)) \setminus E$.

Since E satisfies condition (E1) and $\alpha \notin E$, there is a $k \in I$ such that no $\epsilon \in E$ satisfies

$$\begin{aligned} \epsilon_k &= \alpha_k, \\ \epsilon_i &\geq \alpha_i \text{ for all } i \in I \setminus \{k\}. \end{aligned}$$

We set $\beta := \gamma - \mathbf{e}_k \in D_S$, that is,

$$\begin{aligned} \beta_k &= \tau_k, \\ \beta_i &= \gamma_i \text{ for all } i \in I \setminus \{k\}. \end{aligned}$$

Then $0 \in \Delta_k^S(\tau - \beta) \neq \emptyset$, and Lemma 5.2.15 yields a $\zeta \in \Delta_j^E(\tau - \beta + \alpha) \neq \emptyset$ for some $j \in I$. That is, $\zeta \in E$ with

$$\begin{aligned}\zeta_j &= \tau_j - \beta_j + \alpha_j, \\ \zeta_i &> \tau_i - \beta_i + \alpha_i \text{ for all } i \in I \setminus \{j\}.\end{aligned}$$

We must have $j \neq k$ as otherwise $\epsilon = \zeta$ would contradict the choice of k . Thus,

$$\begin{aligned}\zeta_j &= \alpha_j - 1, \\ \zeta_k &> \alpha_k, \\ \zeta_i &\geq \alpha_i \text{ for all } i \in I \setminus \{j, k\}.\end{aligned}$$

Since $\zeta \in E \subset K_S^0 - (K_S^0 - E) \ni \alpha$ by Lemma 5.2.13.(a), and since $K_S^0 - (K_S^0 - E)$ satisfies condition (E1) by Lemmas 5.2.2 and 4.1.7, we find

$$\alpha > \alpha - \mathbf{e}_j = \min\{\alpha, \zeta\} =: \alpha' \in K_S^0 - (K_S^0 - E).$$

We must have $\alpha' \notin E$ as otherwise condition (E2) satisfied by E applied to $\alpha', \zeta \in E$ would yield an $\epsilon \in E$ contradicting the choice of k . But $\alpha > \alpha' \in (K_S^0 - (K_S^0 - E)) \setminus E$ contradicts the minimality of α . We conclude that $E = K_S^0 - (K_S^0 - E)$ as claimed. \square

Remark 5.2.17. In case $|I| = 1$, there is an easier proof of Proposition 5.2.16. Let $\alpha \in K_S^0 - (K_S^0 - E)$, and set $\beta = \tau$. Since

$$\Delta^S(\tau - \beta) = \Delta(0) \cap S = \{0\} \cap S = \{0\} \neq \emptyset,$$

Lemma 5.2.15 gives

$$\emptyset \neq \Delta^E(\tau - \beta + \alpha) = \Delta(\alpha) \cap E = \{\alpha\} \cap E,$$

and hence $\alpha \in E$. Thus, $E = K_S^0 - (K_S^0 - E)$.

5.3. Duality of fractional ideals.

Lemma 5.3.1. *Let R be an admissible ring and $\mathcal{E}, \mathcal{F} \in \mathfrak{R}_R$. Then $\Gamma_{\mathcal{F}}: \mathcal{E} \subset \Gamma_{\mathcal{F}} - \Gamma_{\mathcal{E}}$ and, in particular, $\Gamma_{\mathcal{C}_{\mathcal{E}}} \subset \Gamma_{\mathcal{E}}$.* \square

The following was proved by D'Anna in case $\mathcal{E} = R$ (see [D'A97, Prop. 1.3]).

Proposition 5.3.2. *Let R be an admissible ring. Then $\mathcal{C}_{\mathcal{E}} = Q_R^{\Gamma_{\mathcal{E}}}$ for any $\mathcal{E} \in \mathfrak{R}_R$. Equivalently, $C_{\Gamma_{\mathcal{E}}} = \Gamma_{\mathcal{C}_{\mathcal{E}}}$.*

Proof. Let $E := \Gamma_{\mathcal{E}}$. By Lemma 5.3.1, $\mathcal{C}_{\mathcal{E}} \subset \{x \in Q_R \mid \nu(x) \geq \gamma^E\} = Q_R^{\gamma^E}$, and $\Gamma_{\mathcal{E}^{\gamma^E}} = \Gamma_{Q_R^{\gamma^E}} = C_E$. Since $\mathcal{E}^{\gamma^E} \subset Q_R^{\gamma^E}$, Corollary 4.2.9 yields $Q_R^{\gamma^E} = \mathcal{E}^{\gamma^E} \subset \mathcal{E}$. With $Q_R^{\gamma^E} \in \mathfrak{R}_R$, it follows that $Q_R^{\gamma^E} = \mathcal{C}_{Q_R^{\gamma^E}} \subset \mathcal{C}_{\mathcal{E}}$, and hence $\mathcal{C}_{\mathcal{E}} \subset \mathcal{E}^{\gamma^E} = Q_R^{\gamma^E} \subset \mathcal{C}_{\mathcal{E}}$. \square

By Proposition 5.3.2, value semigroup ideals commute with conductors in the sense that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{R}_R & \xrightarrow{\mathcal{E} \mapsto \mathcal{C}_{\mathcal{E}}} & \mathfrak{R}_{\overline{R}} \\ \mathcal{E} \mapsto \Gamma_{\mathcal{E}} \downarrow & & \downarrow \mathcal{E} \mapsto \Gamma_{\mathcal{E}} \\ \mathfrak{G}_{\Gamma_R} & \xrightarrow{E \mapsto C_E} & \mathfrak{G}_{\Gamma_{\overline{R}}}, \end{array}$$

We now show the compatibility of value semigroup ideals with dualizing as announced at the beginning of §5. This is easy in case $\mathcal{E} \in \mathfrak{R}_{\overline{R}}$ as the following proposition shows.

Proposition 5.3.3.

- (a) Let S be a good semigroup, and let $E \in \mathfrak{G}_{\bar{S}}$ and $F \in \mathfrak{G}_S$. Then $E = C_E$ and $F - E = C_{F-E}$.
- (b) Let R be an admissible ring, $\mathcal{E} \in \mathfrak{R}_{\bar{R}}$, and $\mathcal{F} \in \mathfrak{R}_R$. Then $\mathcal{E} = C_{\mathcal{E}}$, $\Gamma_{\mathcal{E}} = C_{\Gamma_{\mathcal{E}}}$, and $\Gamma_{\mathcal{F}:\mathcal{E}} = \Gamma_{\mathcal{F}} - \Gamma_{\mathcal{E}}$.

Proof.

- (a) See Definition 4.1.5.
- (b) By Definition 2.1.1.(d), $\mathcal{E} = C_{\mathcal{E}}$ and $\mathcal{F} : \mathcal{E} \in \mathfrak{R}_{\bar{R}}$. Hence, $\Gamma_{\mathcal{E}} = C_{\Gamma_{\mathcal{E}}}$ and $\Gamma_{\mathcal{F}:\mathcal{E}} = C_{\Gamma_{\mathcal{F}:\mathcal{E}}} = C_{\Gamma_{\mathcal{F}} - \Gamma_{\mathcal{E}}} = \Gamma_{\mathcal{F}} - \Gamma_{\mathcal{E}}$ by Theorem 3.1.1.(c), Remark 4.1.6.(b), Proposition 5.3.2 and (a). \square

Lemma 5.3.4. Let R be an admissible ring, and let $\mathcal{E}, \mathcal{F} \in \mathfrak{R}_R$ such that $\mathcal{E} \subset \mathcal{F}$. Then

$$\ell_R(\mathcal{K} : \mathcal{E} / \mathcal{K} : \mathcal{F}) = \ell_R(\mathcal{F} / \mathcal{E}).$$

Proof. See [HK71, Rem. 2.5.(c)]. \square

Theorem 5.3.5. Let R be an admissible ring with canonical ideal \mathcal{K} , and set $K := \Gamma_{\mathcal{K}}$. Let $\mathcal{E}, \mathcal{F} \in \mathfrak{R}_R$ such that $\mathcal{E} \subset \mathcal{F}$. Then

$$d(\Gamma_{\mathcal{F}} \setminus \Gamma_{\mathcal{E}}) = d(\Gamma_{\mathcal{K}:\mathcal{E}} \setminus \Gamma_{\mathcal{K}:\mathcal{F}}) = d(K - \Gamma_{\mathcal{E}} \setminus K - \Gamma_{\mathcal{F}}),$$

and, in particular,

$$\Gamma_{\mathcal{K}:\mathcal{E}} = K - \Gamma_{\mathcal{E}}.$$

Proof. By [HK71, Lem. 2.6], Theorem 4.1.4, Remark 4.2.4.(a) and Proposition 5.2.3 we may assume that R is local. Moreover, by Remarks 2.1.2.(a) and (b) and 4.1.6.(a) and (b), Corollary 5.1.5 and Theorems 5.2.4 and 5.2.7 we may assume that $\mathcal{K} = \mathcal{K}_R^0$, and hence $K := \Gamma_{\mathcal{K}} = K_S^0$.

Proposition 4.2.8 and Lemma 5.3.4 yield

$$d(\Gamma_{\mathcal{K}:\mathcal{E}} \setminus \Gamma_{\mathcal{K}:\mathcal{F}}) = \ell_R((\mathcal{K} : \mathcal{E}) / (\mathcal{K} : \mathcal{F})) = \ell_R(\mathcal{F} / \mathcal{E}) = d(\Gamma_{\mathcal{F}} \setminus \Gamma_{\mathcal{E}}) =: n.$$

There is a composition series in \mathfrak{R}_R (see [AM69, Ch. 6])

$$\mathcal{C}_{\mathcal{E}} = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \cdots \subsetneq \mathcal{E}_l = \mathcal{E} \subsetneq \mathcal{E}_{l+1} \subsetneq \cdots \subsetneq \mathcal{E}_{l+n} = \mathcal{F}.$$

By Corollary 4.2.9 and Proposition 5.3.2, applying Γ yields a chain in \mathfrak{G}_{Γ_R}

$$C_{\Gamma_{\mathcal{E}}} = \Gamma_{\mathcal{E}_0} \subsetneq \Gamma_{\mathcal{E}_1} \subsetneq \cdots \subsetneq \Gamma_{\mathcal{E}_l} = \Gamma_{\mathcal{E}} \subsetneq \Gamma_{\mathcal{E}_{l+1}} \subsetneq \cdots \subsetneq \Gamma_{\mathcal{E}_{l+n}} = \Gamma_{\mathcal{F}}.$$

Dualizing with K we obtain by Remark 4.1.6.(c), Corollary 4.2.9 and Propositions 5.2.16, 5.3.2 and 5.3.3.(b) again a chain in \mathfrak{G}_{Γ_R}

$$(5.4) \quad \begin{aligned} \Gamma_{\mathcal{K}:\mathcal{C}_{\mathcal{E}}} &= \Gamma_{\mathcal{K}} - \Gamma_{\mathcal{C}_{\mathcal{E}}} = K - C_{\Gamma_{\mathcal{E}}} = K - C_{\Gamma_{\mathcal{E}_0}} \supsetneq \cdots \supsetneq K - \Gamma_{\mathcal{E}_l} = K - \Gamma_{\mathcal{E}} \\ &\supsetneq K - \Gamma_{\mathcal{E}_{l+1}} \supsetneq \cdots \supsetneq K - \Gamma_{\mathcal{E}_{l+n}} = K - \Gamma_{\mathcal{F}} \supset \Gamma_{\mathcal{K}:\mathcal{F}}. \end{aligned}$$

Since by Propositions 3.1.8 and 5.2.10 we have $K - \Gamma_{\mathcal{E}_i} \in \mathfrak{G}_S$ for all $i = 0, \dots, l+n$, Proposition 4.2.6 yields $d(K - \Gamma_{\mathcal{E}_i} \setminus K - \Gamma_{\mathcal{E}_{i+1}}) \geq 1$ for all $i = 0, \dots, l+n-1$. Moreover, by Proposition 4.2.8 and Lemma 5.3.4 we have

$$d(\Gamma_{\mathcal{K}:\mathcal{C}_{\mathcal{E}}} \setminus \Gamma_{\mathcal{K}:\mathcal{F}}) = \ell_R(\mathcal{K} : \mathcal{C}_{\mathcal{E}} / \mathcal{K} : \mathcal{F}) = \ell_R(\mathcal{F} / \mathcal{C}_{\mathcal{E}}) = l+n.$$

Therefore, by Lemma 4.2.5 and Equation (5.4) it follows that $d(K - \Gamma_{\mathcal{E}_i} \setminus K - \Gamma_{\mathcal{E}_{i+1}}) = 1$ for all $i = 0, \dots, l+n-1$ and $d(K - \Gamma_{\mathcal{F}} \setminus \Gamma_{\mathcal{K}:\mathcal{F}}) = 0$. By Proposition 4.2.6 the latter is equivalent to the second claim. Lemma 4.2.5 applied to the former equalities yields the first claim:

$$d(K - \Gamma_{\mathcal{E}} \setminus K - \Gamma_{\mathcal{F}}) = \sum_{i=l}^{l+n-1} d(K - \Gamma_{\mathcal{E}_i} \setminus K - \Gamma_{\mathcal{E}_{i+1}}) = n = d(\Gamma_{\mathcal{F}} \setminus \Gamma_{\mathcal{E}}). \quad \square$$

Based on our Definition 5.2.5 of canonical semigroup ideals we can finally generalize D’Anna’s Theorem 5.2.4 to the semilocal case.

Corollary 5.3.6. *Let R be an admissible ring with value semigroup $S := \Gamma_R$. Then for any fractional ideal \mathcal{K} of R the following are equivalent:*

- (i) \mathcal{K} is a canonical ideal of R .
- (ii) $K := \Gamma_{\mathcal{K}}$ is a canonical ideal of S .

Proof. (i) \Rightarrow (ii) By (i) and [HK71, Lem. 2.6], $\mathcal{K}_{\mathfrak{m}}$ is a canonical ideal of $R_{\mathfrak{m}}$ for all $\mathfrak{m} \in \text{Max}(R)$. By Lemma 2.1.5 there exists an $x_{\mathfrak{m}} \in Q_{R_{\mathfrak{m}}}^{\text{reg}}$ such that $R_{\mathfrak{m}} \subset x_{\mathfrak{m}}\mathcal{K}_{\mathfrak{m}} \subset \overline{R_{\mathfrak{m}}}$. By Proposition 5.1.3, $x_{\mathfrak{m}}\mathcal{K}_{\mathfrak{m}}$ is a canonical ideal with $\Gamma_{x_{\mathfrak{m}}\mathcal{K}_{\mathfrak{m}}} = \nu(x_{\mathfrak{m}}) + \Gamma_{\mathcal{K}_{\mathfrak{m}}}$. Then (ii) follows from Theorems 3.2.2 and 5.2.7 and Proposition 5.2.3.

(ii) \Rightarrow (i) Let $\mathcal{E} \in \mathfrak{R}_R$ with $E := \Gamma_{\mathcal{E}}$. Then $\mathcal{E} \subset \mathcal{K} : (\mathcal{K} : \mathcal{E})$, and $\Gamma_{\mathcal{K} : (\mathcal{K} : \mathcal{E})} = K - (K - E) = E$ by assumption and Theorems 5.3.5 and 5.2.7.(iii). By Corollary 4.2.9, $\mathcal{E} = \mathcal{K} : (\mathcal{K} : \mathcal{E})$ proving (i). \square

Definition 5.3.7. We call a good semigroup S *symmetric* if S is a canonical ideal of S .

Corollary 5.3.8. *Let R be an admissible local ring. Then R is Gorenstein if and only if Γ_R is symmetric.*

Proof. This follows immediately from Corollary 5.3.6. \square

REFERENCES

- [AM69] M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969. MR 0242802 (39 #4129) 5.3
- [BDF00] V. Barucci, M. D’Anna, and R. Fröberg, *Analytically unramified one-dimensional semilocal rings and their value semigroups*, J. Pure Appl. Algebra **147** (2000), no. 3, 215–254. MR 1747441 (2001g:13054) 1, 3.1, 3.2, 4.1, 4.1.9, a, 5.2
- [BH93] Winfried Bruns and Jürgen Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993. MR MR1251956 (95h:13020) 3.3, 5.1.2
- [Bou61] N. Bourbaki, *Éléments de mathématique. Fascicule XXVII. Algèbre commutative. Chapitre 1: Modules plats. Chapitre 2: Localisation*, Actualités Scientifiques et Industrielles, No. 1290, Herman, Paris, 1961. MR 0217051 (36 #146) 2.1
- [CDK94] A. Campillo, F. Delgado, and K. Kiyek, *Gorenstein property and symmetry for one-dimensional local Cohen-Macaulay rings*, Manuscripta Math. **83** (1994), no. 3-4, 405–423. MR 1277539 (95d:13028) 1, 2.2, 3.3, 4.1
- [D’A97] Marco D’Anna, *The canonical module of a one-dimensional reduced local ring*, Comm. Algebra **25** (1997), no. 9, 2939–2965. MR 1458740 (98h:13031) 1, 3.1.5, 3.1, 3.3, 3.3.5, 4.2, 4.2, 4.2, 4.2, 5, 5.2, 5.2, 5.2, 5.3
- [Del87] Félix Delgado de la Mata, *The semigroup of values of a curve singularity with several branches*, Manuscripta Math. **59** (1987), no. 3, 347–374. MR 909850 (89f:14022) 1, 3.3.5, 4.1, 5
- [Del88] ———, *Gorenstein curves and symmetry of the semigroup of values*, Manuscripta Math. **61** (1988), no. 3, 285–296. MR 949819 (89h:14024) 3.1, 3.1.5, 3.3.5, 4.1, 4.1, 5
- [Dic13] Leonard Eugene Dickson, *Finiteness of the Odd Perfect and Primitive Abundant Numbers with n Distinct Prime Factors*, Amer. J. Math. **35** (1913), no. 4, 413–422. MR 1506194 4.1.1
- [Gar82] Arnaldo García, *Semigroups associated to singular points of plane curves*, J. Reine Angew. Math. **336** (1982), 165–184. MR 671326 1
- [HK71] Jürgen Herzog and Ernst Kunz (eds.), *Der kanonische Modul eines Cohen-Macaulay-Rings*, Lecture Notes in Mathematics, Vol. 238, Springer-Verlag, Berlin-New York, 1971, Seminar über die lokale Kohomologietheorie von Grothendieck, Universität Regensburg, Wintersemester 1970/1971. MR 0412177 (54 #304) 2.1, b, 5.1, 5.1.2, 5.1, 5.1, 5.3, 5.3, 5.3
- [Jäg77] Joachim Jäger, *Längenberechnung und kanonische Ideale in eindimensionalen Ringen*, Arch. Math. (Basel) **29** (1977), no. 5, 504–512. MR 0463156 (57 #3115) 1, 2.1, 5

- [Kun70] Ernst Kunz, *The value-semigroup of a one-dimensional Gorenstein ring*, Proc. Amer. Math. Soc. **25** (1970), 748–751. MR 0265353 (42 #263) [1](#)
- [KV04] K. Kiyek and J. L. Vicente, *Resolution of curve and surface singularities*, Algebras and Applications, vol. 4, Kluwer Academic Publishers, Dordrecht, 2004, In characteristic zero. MR 2106959 (2005k:14028) [2.1](#), [a](#), [2.2](#), [a](#), [b](#), [c](#), [2.2](#), [2.2](#), [2.2.3](#), [3.1](#), [3.1](#), [3.1](#), [3.1](#), [b](#), [3.3](#), [3.3](#), [3.3.5](#)
- [KW84] Ernst Kunz and Rolf Waldi, *Über den Derivationenmodul und das Jacobi-Ideal von Kurvensingularitäten*, Math. Z. **187** (1984), no. 1, 105–123. MR 753425 (85j:14033) [3.3.5](#)
- [LJ73] Monique Lejeune-Jalabert, *Sur l'équivalence des courbes algébroides planes. Coefficients de Newton. Contribution à l'étude des singularités du point de vue du polygone de Newton*, Université de Paris VII, Paris, 1973, Thèse présentée à l'Université Paris VII pour obtenir le grade de Docteur ès-Sciences Mathématiques, Partiellement en collaboration avec Bernard Teissier. MR 0379895 (52 #800a) [1](#)
- [Mat73] Eben Matlis, *1-dimensional Cohen-Macaulay rings*, Lecture Notes in Mathematics, Vol. 327, Springer-Verlag, Berlin-New York, 1973. MR 0357391 (50 #9859) [2.2](#)
- [Mat89] Hideyuki Matsumura, *Commutative ring theory*, second ed., Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989, Translated from the Japanese by M. Reid. MR 1011461 (90i:13001) [2.1](#)
- [Nag62] Masayoshi Nagata, *Local rings*, Interscience Tracts in Pure and Applied Mathematics, No. 13, Interscience Publishers a division of John Wiley & Sons New York-London, 1962. MR 0155856 (27 #5790) [3.3](#), [4.2](#)
- [Pol15a] Delphine Pol, *Logarithmic residues along plane curves*, C. R. Math. Acad. Sci. Paris **353** (2015), no. 4, 345–349. MR 3319132 [1](#)
- [Pol15b] ———, *On the values of logarithmic residues along curves*, arXiv.org **1410.2126** (2015). [1](#)
- [Zar86] Oscar Zariski, *Le problème des modules pour les branches planes*, second ed., Hermann, Paris, 1986, Course given at the Centre de Mathématiques de l'École Polytechnique, Paris, October–November 1973, With an appendix by Bernard Teissier. MR 861277 (88a:14031) [1](#)

PH. KORELL, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KAISERSLAUTERN, 67663 KAISERSLAUTERN, GERMANY

E-mail address: korell@mathematik.uni-kl.de

L. TOZZO, DIPARTIMENTO DI MATEMATICA, UNIVERSITA' DEGLI STUDI DI GENOVA, VIA DODECANESO 35, 16154 GENOVA, ITALY

Current address: L. Tozzo, Department of Mathematics, University of Kaiserslautern, 67663 Kaiserslautern, Germany

E-mail address: tozzo@mathematik.uni-kl.de

M. SCHULZE, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KAISERSLAUTERN, 67663 KAISERSLAUTERN, GERMANY

E-mail address: mschulze@mathematik.uni-kl.de